Modeling the process of information relay through inter-vehicle communication

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Abstract

In a new paradigm of the decentralized traffic information system as a recent thrust in the Intelligent Transportation Systems (ITS), vehicles form ad hoc mobile networks, and information may be propagated between vehicles through wireless communication with a short transmission range. Fundamental to the system design is effective information propagation. In this paper, we study information propagation along a traffic stream on which presence of equipped vehicles follows an independent homogeneous Poisson process. We define a relay process in which only the furthest equipped vehicle within each transmission range continues the relay, and model it as a transient Markov process. We present closed form formulas for the expected value and variance of propagation distance in the case without transmission delay. We also study the expected number of relays and the expected propagation distance in the case with transmission delay. The results make transparent the relationship between propagation distance, equipped vehicle density and transmission range. In addition, we study the probability distribution of propagation distance, and find that the Gamma distribution could be used as a good practical means of approximation especially when the number of equipped vehicles is large within a transmission range. The Gamma-like behavior is also observed on heterogeneous traffic. It is noted that the relay process has many other applications as well.

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1. Introduction

A recent thrust in the Intelligent Transportation Systems (ITS), inter-vehicle communication has witnessed an increasingly growing effort in recent years. Enabled by the well-developed wireless communication technology, inter-vehicle communication (IVC) with short transmission range has been considered practically feasible to support a decentralized real time traffic information system that 'floats' on the traffic stream, resembling an 'internet on the road' (Jin and Recker, 2006). In contrast to the centralized traffic information system in which traffic information is collected and disseminated through an information center such as ADVANCE
(see Boyce et al., 1994), in this paradigm, vehicles are equipped with communication capability, and act as 'servers on the internet', capable of generating, receiving, transmitting and managing information (see CarTALK, 2000; FleetNet, 2003). Information can be propagated in a relay process via these vehicles along the stream of traffic.

As traffic is a random phenomenon, information relay may die out any moment for the lack of equipped vehicles within a transmission range. Fundamental to developing and deploying a decentralized information system with inter-vehicle communication is effective information propagation: how far can information be effectively and reliably propagated along a traffic stream? What factors contribute to effective information propagation? These questions are signified in Yang (2003) and Jin and Recker (2006) as they are critical to the successful design of decentralized traffic information systems with such parameters as transmission range and percentage of vehicles equipped (often referred to as market penetration rate). In this paper, we specifically address the relationship between equipped traffic density, transmission range and information propagation distance.

Such a decentralized real time traffic information system has a significant potential for application. It could dramatically enrich the traffic information with which drivers perform their driving functions. In particular, drivers could become significantly less myopic in driving, therefore making the traffic more resilient to disturbances. We would refer interested readers to Yang (2003) for more detailed technical and practical discussions. To summarize, such a real time decentralized information system may serve both as a significant enhancement and as a complement to the centralized traffic information system as it offers an extra capacity of generating, capturing, accommodating and managing instantaneous information of interest to drivers within small scopes or during short transient time periods. Ideally, this decentralized information system will develop into an autonomous, self-organizing information network for effective management of interactions among intelligently informed vehicles, roadways, stations and consumers (Jin and Recker, 2006). While the decentralized traffic information system with inter-vehicle communication has many technical and practical challenges, it has been considered as a promising next generation ITS in addressing quickly exacerbated urban traffic congestion and related air pollution problems in the contemporary society.

Albeit attention to application of inter-vehicle communication has been paid for years, the literature on information propagation remains relatively scant. Much of the relevant literature is on radio transmission in the area of wireless communication. As one of the early efforts, Takagi and Kleinrock (1984) study a radio transmission problem which assumes that distribution of transmission terminals within a geographic region follows a two-dimensional homogeneous Poisson distribution. But the authors only consider expected one step progress towards the direction of interest. Recently, Jin and Recker (submitted for publication) present a new model for multihop connectivity between vehicles, in which a thorough review of literature on connectivity between a pair of nodes is available. Ziliaskopoulos and Zhang (2003) study information propagation based on traffic flow in opposite directions. Most relevantly, Yang (2003) conducts extensive micro-simulations on computer, and many empirical insights to information propagation become available. Jin and Recker (2006) complement that study by developing analytical methods. Specifically, in addressing the effective propagation distance, Jin and Recker (2006) present a recursive model to calculate the probability of covering each vehicle spot (in the discrete case) along a traffic stream. Their recursive method offers a numerical way to the probability distribution of instantaneous propagation distance. However, an analytical relationship between equipped vehicle density, transmission range, and propagation distance remains unclear. Closed form formulas to analytically calculate expected value, variance and probability distribution of propagation distance have yet to be developed, which would be much beneficial to both practice and research. Although Cheng and Robertazzi (1989) in their one-dimensional case study almost the same characteristics as ours on a mobile radio network and present closed form formulas for the expected distance and expected number of cycles (the same as relays in this paper), their results often resort to intermediate assumptions and approximations, and therefore fall short of accuracy and rigorosness.

In this paper, we provide a recursive formula for the probability of successful propagation as a continuous alternative to the counterpart in Jin and Recker (2006). In addition, we make the following contributions:

- We identify a relay process that decides the propagation distance, and model it as a transient Markov process. The relay process defined has many practical applications as will be explained at the end of this paper.
We develop closed form formulas for the expected distance of information propagation with instantaneous transmission, and its variance. We further study the expected number of relays and the expected propagation distance in the case with transmission delay.

We study probability distribution of propagation distance, and propose closed form approximation to it with Gamma distribution. Numerical examples indicate that the Gamma distribution could be a good practical means of approximation. In addition, the Gamma-like behavior is also observed on the general traffic. We hope that the results here may add to the basic tool kit available for designing and implementing a decentralized traffic information system through inter-vehicle communication.

We first present the problem of study as follows.

1.1. Problem statement

A traffic stream is present on a line of infinite length. Presence of communication capable vehicles (referred to as vehicles later for simplicity) follows an independent homogeneous Poisson process with a constant point intensity \( l \) along the stream of traffic. Information may be transmitted in a direction from one vehicle to another, called a relay. The transmission range of each vehicle is \( L \). If there is no communication capable vehicle within range \( L \), the information dies out. In this case, the propagation distance equals to the distance of last communication vehicle to the origin of information. We study how far information may be propagated on the traffic stream in the direction.

In what follows, we present our model in the discrete case in Section 2 where a relay process is defined and modeled as a transient Markov process. We hope that the discrete model will help illustrate our basic ideas, and facilitate discussions in the continuous case. In Section 3, we study the expected propagation distance and its variance in the case with instantaneous transmission where vehicles can be considered motionless. In Section 4, the probability distribution of propagation distance and the Gamma approximation are studied. In Section 5, the expected number of relays and the expected propagation distance are presented in the case with transmission delay where vehicle displacement during each relay is considered. We conclude this paper in Section 6 in which further discussion of information propagation over heterogeneous traffic is conducted.

2. Discrete modeling of the relay process

From Sections 2–4, we consider instantaneous information transmission in which transmissions take no time compared to the moving speed of vehicles. In Section 5, we study separately information propagation with transmission delay.

In order to facilitate the presentation, we start with the discrete case of the problem.

In the discrete case, we assume that within each transmission range \( L \), there are \( L \) units of consecutive sections, each of which is capable of accommodating one and only one vehicle. Vehicles are located in the sections, and no vehicle spans two or more sections. The sections are indexed with numbers from 1 to \( L \) in the propagation direction of interest. The presence probability at each section is \( l \). Obviously, \( l < 1 \).

At a relay, if the furthest vehicle in range \( L \) takes position at section \( k \), no other vehicles within transmission range is able to propagate the information to a further distance than this vehicle at section \( k \). Therefore, in deciding the maximum propagation distance, only the furthest vehicle in the range \( L \) of each relay is considered. This resembles the ‘most forwarded within-range’ (MFR) concept first seen in Takagi and Kleinrock (1984) and later adopted in Jin and Recker (2006). Furthermore, in the relay that follows, the distance between sections \( k \) and \( L \) has been verified by this relay to have no presence of equipped vehicles. This distance is referred to as void distance in the next relay. Fig. 1 shows three consecutive relays, each starting with a void distance that immediately follows the communication vehicle. The starting void distance of a relay is verified by the relay that immediately proceeds it.

We define a relay process as follows.

**Definition 1.** A relay process is defined as a set of consecutive relays in a certain direction via inter-vehicle communication that has the following properties:
(a) The relay process takes place in one direction on a line of infinite length. The presence probability of a vehicle within any section on this line is $\mu$, and the no-presence probability is $1 - \mu$. Presence probabilities of non-overlapping sections are independent.

(b) Each relay has a fixed transmission range $L$. Only the furthest vehicle within range $L$ continues the relay process.

(c) The relay process terminates when there is no vehicle within range $L$.

A relay defined here is equivalent to the ‘cycle’ in Cheng and Robertazzi (1989). In line with the MFR concept, the following holds.

Proposition 1. The relay process determines the information propagation distance via inter-vehicle communication.

Therefore, we only need to study the relay process for propagation distance.

In a relay process, each relay is characterized by a beginning void distance verified by the relay that immediately proceeds it. (The void distance of the initial relay may be considered given though.) Here we denote the state of each relay by $s(t)$, where $t$ is its void distance to start with. It is easily seen that the relay process is one that transits between the states $s(t)$ where $0 \leq t \leq L$. Obviously, state $s(L)$ implies failure to further propagate as there is no vehicle within the range $L$. We say that $s(L)$ can only transit into itself. The relay process is a Markov process as the transition probability from state $s(t_1)$ to state $s(t_2)$ only depends on state $s(t_1)$ irrespective of the history prior to $s(t_1)$. The following is an example to show the transition from state $s(3)$ to $s(5)$ when $L = 10$.

In order to transit into $s(3)$, there must be a distance of five sections at the end of the transmission range $L$ without presence of vehicles. The probability equals to that of having no vehicle in the last five sections of the transmission range and having one vehicle at the 6th last section, which is $\mu(1 - \mu)^5$ from state $s(t)$ where $L - t > 5$, and 0 otherwise.

Note that the transition probability from any state $s(t)$ to state $s(L)$ is $\mu^{L-t}$, which is always positive. Specially, the transition probability from $s(L)$ to itself is 1.0.

To summarize the above discussion, we have the following result.

Theorem 1. A relay process is a Markov process. It has the following properties:

(a) Each relay starts with a state $s(t)$ characterized by a void distance $t$. The void distance is the first $t$ sections within range $L$. The presence probability of vehicles is zero within the void sections, and $\mu$ in any other section within the range $L$. Presence probabilities of non-overlapping sections are independent.

(b) There is a finite set of states
$$S = \{s(t) : 0 \leq t \leq L, \ t \text{ and } L \text{ are integers}\}.$$

(c) The transition probability from state $s(t_1)$ to state $s(t_2)$ is $\mu(1 - \mu)^{t_2 - t_1}$ when $t_2 < L - t_1$, and 0 when $L - t_1 \leq t_2 < L$. It is $(1 - \mu)^{L-t}$ from state $s(t)$ to state $s(L)$ when $0 \leq t \leq L$.

Specially, due to property (c), the probability is 0 to transit from state $s(L)$ into any other state $s(t)$ where $t < L$. Theorem 1 outlines the basic rational for the study in this paper.

Furthermore, we have the following result.

\[\text{Fig. 1. A relay process as information propagated rightwards.}\]
Proposition 2. The relay process has a finite expected propagation distance.

Proof. Note that all the states except for \( s(L) \) are transient. A transient state is one that there always exists a positive probability never to re-enter it in the future. In a Markov process, a transient state will be visited only finitely often expectedly (Ross, 1997, Chapter 4). State \( s(L) \) is recurrent, but does not propagate information further. Therefore, the total expected propagation distance is bounded by the product between transmission range \( L \) (limit to each forward movement) and the expected number of visits to the transient states. This finishes the proof. \( \square \)

We denote by \( X(t) \) the propagation distance starting at state \( s(t) \), and \( x(t) = E[X(t)] \). Again, \( X(t) \) is the distance from the origin of information to the last vehicle that fails to propagate further.

Note that transition between states leads to information propagated forward. As an example, start with \( s(L - 1) \), which means that there are \( L - 1 \) void sections from the origin of information. There are two outcomes: transit to \( s(L) \) and \( s(0) \) at probabilities \((1 - \mu)\) and \( \mu \), respectively. The first outcome means that section \( L \) from the origin of information does not have communication capable vehicle. Therefore, it fails to propagate further and gives zero distance forward in this case. The new verified void distance becomes \( L \), and the new state is \( s(L) \). The second case means that there is a vehicle at section \( L \). And the propagation continues from this new vehicle. Note that there is no verified void distance following this new vehicle (this vehicle is at the end of a transmission range). As a result, the new state is \( s(0) \). Therefore, the total propagation distance from the origin of information becomes \( L + X(0) \) in the second case, where \( X(0) \) represents expected additional propagation distance.

In general, if the state transits into \( s(i) \) from \( s(t) \), we have

\[
X(t) = L - i + X_1(i), \quad \text{conditional on the next state being } s(i),
\]

where \( X(t) \) and \( X_1(i) \) are i.i.d. Therefore, conditional on the outcomes, we have \( E[X(t)] = E[E[X(t)|i]] \), and specifically \( X(L) = 0 \). Equivalently we have the following recursion in light of Theorem 1 to show the transitional relationships

\[
x(k) = \sum_{i=0}^{L-k-1} \mu(1 - \mu)^i[L - i + x(i)] \quad \forall 0 \leq k \leq L - 1
\]

and \( x(L) = 0 \).

The above array has \( L \) equations with \( L \) variables. As the expected propagation distances are unique, the solution to the above equations must be unique. In fact, there is an algorithm (see Wang, 2005) with only 2\( L \) steps to obtain the solution.

Of interest to communication systems design is the number of forward relays in a relay process. We denote by \( Y(t) \) the number of relays with positive forward distance starting at state \( s(t) \), \( y(t) \) as its expected value, i.e., \( y(t) = E[Y(t)] \). Note here that \( Y(t) \) is the same as the number of hops in Jin and Recker (2006).

Correspondingly, we have

\[
y(k) = \sum_{i=0}^{L-k-1} \mu(1 - \mu)^i[1 + y(i)] \quad \forall 0 \leq k \leq L - 1
\]

and \( y(L) = 0 \).

Similar to (1), solution to Eq. (2) is unique.

With discussions in the discrete case, we are now ready to move forward to the continuous case. In this way, we can develop closed form formulas of information propagation for the ease of use. We devote the remainder of this paper to discussions in the continuous case.

3. Continuous modeling of the relay process

From now on, we assume that vehicles are just points on a line. Point assumption of vehicles in traffic study is popular. Striking proximity of our analytical result to results from simulation in other research papers will show that the point assumption is adequate in this study.
3.1. Expected propagation distance of the relay process

Clearly, there is a close connection between the discrete and continuous cases. When the number of sections in a transmission range \( L \) tends to infinity, it naturally gives rise to the following assumption (Details see Bain and Engelhardt, 1991, Chapter 3, Theorem 3.2.3).

**Assumption 1.** The presence of vehicles on the line of information propagation follows an independent homogeneous point Poisson process whose point intensity is denoted by \( \mu \).

Without much notational abuse, we still use the same notation as in the previous section except that all the discrete measures become continuous throughout the remainder of the paper.

Specifically, the state space becomes
\[
S = \{s(t) : 0 \leq t \leq L, \ t \text{ and } L \text{ are real numbers}\}.
\]

And the transition probability density function from \( s(t_1) \) to \( s(t_2) \) is \( \mu e^{-\mu t} \) with \( t_2 \leq L - t_1 \), and is 0 when \( L - t_1 < t_2 < L \). The probability to transit into \( s(L) \) is \( e^{-\mu(L-t_1)} \). Corresponding to Eq. (1), we have the following counterpart:
\[
x(l) = \int_0^{L-l} \mu(L - \tau + x(\tau))e^{-\mu \tau}d\tau.
\]

Eq. (3) is explained as follows. The presence probability of the first vehicle at location \( \tau \) to the end of the transmission range \( L \) is \( \mu e^{-\mu \tau}d\tau \). If the first vehicle takes place at \( \tau \), the state transits into state \( s(\tau) \) with an expected additional propagation distance \( x(\tau) \) following a forward distance \( L - \tau \).

Eq. (3) implies the following result.

**Theorem 2.** The expected distance that information is propagated starting at state \( s(t) \) is given by
\[
x(t) = t - L - \frac{1}{\mu} e^{\mu t} + 1 + \frac{1}{\mu} e^{\mu L}, \quad 0 \leq t \leq L.
\]

**Proof.** See Appendix A.2. □

One can easily verify that Eq. (4) satisfies (3).

Eq. (4) makes transparent the relationship between vehicle density, transmission range and expected propagation distance. It is clear that \( x(t) \) is an increasing function of both \( \mu \) and \( L \).

**Observations**

\( x(t) \) is a concave function of the void distance \( t \). As seen in Fig. 2, when \( t \) is within about 30% of its transmission range \( L \), the distance \( x(t) \) does not decrease significantly with \( t \). However, when the void increases from 70% to 100% of \( L \), the expected propagation distance drops sharply. This observation indicates that the propagation distance is not sensitive to \( t \) at small void distances.

![Fig. 2. The expected propagation distance at \( \mu = 2.0 \) and \( L = 1.0 \).](image-url)
Table 1 demonstrates the expected propagation distance at various combinations of \( l \) and \( L \). The deciding factor to propagation distance is \( l L \), the number of vehicles within each transmission range.

### 3.2. Variance of propagation distance

Variance is an important measure of reliability.

**Theorem 3.** The variance of information propagation distance is given by

\[
V(X(t)) = -\frac{c_{X}^{2\mu}}{\mu^2} - \frac{2L \sigma_{X}^{2\mu} L}{\mu} + \frac{c_{X}^{2\mu L}}{\mu^2} + \frac{2t}{\mu} \sigma_{X}^{\mu}.
\]

The proof is provided in Appendix A.3.

As a special case, we have

\[
V(X(0)) = \frac{c_{X}^{2\mu L} - 2\mu L \sigma_{X}^{2\mu L} - 1}{\mu^2}.
\]

Indicated in **Fig. 3**, the standard deviation of \( X(t) \) remains almost constant when the void distance \( t \) is below 60\% of the transmission range \( L \).

We have the following result.

**Corollary 1.** When \( \mu L \gg \mu t \), the following holds:

\[
\lim_{\mu L \to \infty} \frac{\sqrt{V(X(t))}}{E[X(t)]} = 1.
\] (5)
Corollary 1 becomes obvious considering the fact that $e^{2\mu L}$ represents the dominant term in both $x(t)$ and $V(X(t))$ when $\mu L \gg \mu t$. Eq. (5) stands for coefficient of variation.

As a matter of fact, in order to have a ratio in (5) close to 1, the number of vehicles $\mu L$ within a transmission range does not have to be large. Table 2 shows ratios at fairly small $\mu L$ values when $\mu t = 0$. We can find that the expected transmission distance and the standard deviation at $\mu t = 0$ are very close even when $\mu L = 2.4$. Table 2 indicates that the coefficient of variation of the propagation distance cannot be reduced to below 1.0 by changing either the transmission range or density of equipped vehicles. It reveals a volatile aspect of inter-vehicle communication especially when $\mu L$ is small.

Furthermore,

**Corollary 2.** When $t \to L$, the following holds:

$$
\lim_{t \to L} \frac{E[X(t)]}{\sqrt{V(X(t))}} = 0.
$$

**Proof.** As $t \to L$, we have $E[X(t)] \to 0$ and $\sqrt{V(X(t))} \to 0$. In this case, taking derivative of both the numerator and denominator of (6) with respect to $t$ gives rise to a ratio 0 according to $L'$ Hospital’s Rule (see, for example, Thomas and Finney, 1988, Chapter 3).

Corollary 2 shows that as the void distance approaches the length of a full transmission range $L$, the propagation becomes rapidly volatile and instable.

### 4. Closed form approximation to probability distribution of propagation distance

#### 4.1. Probability distribution of propagation distance

Denote by $F(t, x)$ the probability for $X(t) \geq x$ starting with state $s(t)$, where $x$ is a given distance from the origin of information. $F(t, x)$ represents probability of successful propagation beyond point $x$. We have the following result.

**Theorem 4**

$$
F(t, x) = \begin{cases}
\int_0^{t-L} e^{-\mu \tau} F(\tau, x - (L - \tau)) d\tau, & L < x, \\
1 - e^{-\mu(L-t)} - (x-t)e^{-\mu L}, & t < x \leq L, \\
1 - e^{-\mu(L-t)}, & 0 < x \leq t \neq 0, \\
1, & x = 0.
\end{cases}
$$

**Proof.** The result at $x > L$ conditions on the first relay taking place at point $L - \tau$ from the origin of information. $\tau$ represents the distance of first relay to the end of first transmission range $L$. This implies that the first relay transits into state $s(\tau)$. Therefore the transition probability according to continuous Poisson process
is \( \mu e^{-\mu t} dt \). Conditional on the starting state \( s(\tau) \) at point \( L - \tau \), the probability of propagating beyond point \( x \) is \( F(\tau, x - (L - \tau)) \). In addition, as the starting state is \( s(t) \), \( \tau \) can only have a value within the range \([0, L - t]\). In this case, the propagation probability function becomes recursive.

When \( t < x \leq L \), the probability of successful propagation can be represented by the following:

\[
F(t, x) = 1 - e^{-\mu(L-x)} + \int_{L-x}^{L-t} \mu e^{-\mu t} (1 - e^{-\mu(L-t)}) dt,
\]

where \( 1 - e^{-\mu(L-x)} \) represents the presence probability within the range \([x, L]\), which means that the first relay is beyond point \( x \). The second term refers to the case in which the first relay does not go further than point \( x \), which implies that there is no presence of equipped vehicle within \([x, L]\). In this case, in order to propagate beyond \( x \), there must be two conditions satisfied simultaneously. One is that a vehicle be present within \([t, x]\) (or equivalently, \( \tau \in [L - x, L - t]\)) in order to take the first relay. The other is that there must be a second relay beyond \( L \). If the vehicle in the first relay is present at \( \tau \) to the end of the first transmission range \( L \) whereas \( L - t > \tau > L - x \), then there must be vehicle(s) present in the range \([L, 2L - \tau]\) in order for the information to be propagated beyond \( x \). Be aware that the presence probability within the range \([L, 2L - \tau]\) is \( 1 - e^{-\mu(L-\tau)} \).

The results when \( x \) is smaller are obvious. □

Theorem 4 shows that within the range \( t < x \leq L \), the probability density function of \( x \) is a constant.

Theorem 4 enables us to recursively calculate the success probability beyond any point \( x \).

### 4.2. Approximation with Gamma distribution

There are practical benefits of having a closed form probability distribution of propagation distance or its closed form approximation. A closed form formula makes practically convenient reliability analysis of information propagation. In this section, we propose to use Gamma distribution to approximate \( F(\cdot) \).

A Gamma distribution yields the same expected value and variance as those calculated in Section 3 by setting the parameters \( \theta(\mu, L) = \frac{L}{f(X(0))} \), and \( k(\mu, L) = \frac{L^2 f(0)}{f(0)} \), where \( \theta \) and \( k \) are two parameters of Gamma distribution. Introduction to Gamma distribution is in Appendix A.1.

We set \( t = 0 \) for the following examples as \( s(0) \) is usually the most concerned starting state.

#### 4.2.1. Case 1: large \( \mu L \)

Empirically, when \( \mu L \) is larger than 2.50, Gamma distribution represents a good approximation to \( F(t, x) \). Specially, when \( k \to 1 \), Gamma distribution becomes exponential distribution. The mean and variance of the exponential distribution are both \( \theta \). According to Table 2, we find that the expected value and standard deviation of propagation distance are very close to each other even when starting at small \( \mu L \) values. We therefore propose to use exponential distribution as the approximation when the mean and standard deviation are

![Gamma approximation of successful propagation at \( \mu = 5.6, L = 0.5 \) and \( t = 0.0 \).](attachment:image.png)
approximately the same. Therefore, by setting \( \theta = E[X(0)] \) and \( k = 1 \), we can approximate to achieve almost the same mean and standard deviation. An advantage of using exponential distribution is its simple form and therefore ease for use.

Figs. 5 and 6 demonstrate the close approximations.

4.2.2. Case 2: medium \( \mu L \)

As in Fig. 6, the Gamma function appears to be a good fit to the true distribution when \( \mu L \) is as small as 1.0.

4.2.3. Case 3: small \( \mu L \)

In this case, our discussion only has theoretical interest as an effective information propagation does not exist at very small \( \mu L \) values. However, it might illustrate why Gamma approximation does not work as well as in the case of large \( \mu L \).

At small \( \mu L \) values, usually the expected propagation distance is very small compared to the transmission range \( L \). Therefore, Theorem 4 in the case of \( t < x \leq L \) may be practically sufficient for successful propagation evaluation.

Fig. 7 shows approximation of Gamma function to \( F(x) \) within a full range \( L \). Note that the expected propagation distance \( x(0) = 0.0997 \) in this case.

Note that the Gamma approximation in the range \([0, \delta]\), where \( \delta \) is small, shows large errors. The reason is that \( F(\cdot) \) always has a positive value not to propagate the information at all while Gamma distribution gives zero probability of no propagation. This causes the Gamma curve to be systematically below \( F(\cdot) \) at large \( x \) values in order to maintain the same expected value and variance as \( F(\cdot) \). The no-propagation probability of
\( F(\cdot) \) is reduced when \( \mu L \) increases. Therefore, we have a conjecture that Gamma distribution represents the limiting distribution of \( F(\cdot) \) when \( \mu L \to \infty \). After all, the errors at \( x \) values close to zero are not critical as reliability analysis concerns more about much larger \( x \) values.

By comparing Figs. 4 and 5 with Fig. 2 in Jin and Recker (2006), the proximity of the curves shows that the point assumption of vehicles is adequate for this study.

5. Propagation distance with transmission delays

In this section, we calculate the expected distance of information relay when there is a transmission delay. When one imagines that the processor on a vehicle might receive many requests at the same time for further transmission, a delay in processing each request could take place. In the following, we develop analytical formulas to calculate the expected propagation distance in this case. We assume that during each delay, the traffic mix and the relative location of vehicles have not been changed within its void distance.

First, we calculate the expected number of relays in a relay process.

5.1. Expected number of relays

Similar to \( x(t) \), the following recursion for \( y(t) \) is obvious:

\[
y(t) = \int_0^{L-t} \mu(1 + y(t))e^{-\mu t} \, dt .
\]

(7)

\[\text{Theorem 5.} \quad \text{The expected number of relays in a relay process is given as follows:}
\]

\[
y(t) = \begin{cases} 
\frac{\mu e^{\beta t}}{\beta e^{\beta t} - e^{\beta t}} e^{\mu t} + \frac{2\mu e^{\beta t}}{\beta e^{\beta t} - e^{\beta t}} e^{\mu t} - 1, & \mu L > \ln 4, \\
\frac{2\mu e^{\beta t}}{\beta e^{\beta t} - e^{\beta t}} e^{\mu t} - 1, & \mu L = \ln 4, \\
e^{\beta t} (c_1 \cos(\beta t) + c_2 \sin(\beta t)) - 1, & \mu L < \ln 4,
\end{cases}
\]

where

\[
\beta = \frac{\mu}{2} \sqrt{4e^{-\mu L} - 1},
\]

\[
c_1 = \frac{-\frac{2\mu}{\mu} e^{-\frac{\mu}{2} \cos(\beta L)} - e^{-\frac{\mu}{2} \sin(\beta L)}}{2e^{-\frac{\mu}{2} \sin(\beta L)} - \frac{2\mu}{\mu}}.
\]
Proof. Eq. (7) implies the above result. The proof is similar to that for Theorem 2, is therefore omitted here. A detailed proof is also available in Wang (2005).

Different from Cheng and Robertazzi (1989) who present a formula for the expected number of broadcast cycles (the same as relays in this paper) in its one-dimensional model, the result above is accurate and rigorous. □

5.2. Expected propagation distance with transmission delays

Theorem 6. Assume that the void distance does not change during a transmission delay. If the expected transmission delay at each vehicle corresponds to a displacement of \( \Delta \) along the direction of propagation, then the expected propagation distance is as follows:

\[
x(t) = t - L - \frac{1}{\mu}e^{\mu t} + \frac{1}{\mu}e^{\mu L} + \Delta \cdot (y(t) + 1).
\]

Proof. The recursion in this case becomes

\[
x(t) = \int_0^{L-t} \mu(L - \tau + x(\tau))e^{-\mu \tau}d\tau + \Delta.
\] (8)

The solution \( x(t) \) to Eq. (8) comprises of two parts. The first part \( x_1(t) \) makes the following hold:

\[
x_1(t) = \int_0^{L-t} \mu(L - \tau + x_1(\tau))e^{-\mu \tau}d\tau,
\]

which is equivalent to Eq. (3). Therefore, \( x_1(t) \) is the expected propagation distance without transmission delay.

The second part \( x_2(t) \) makes the following hold:

\[
x_2(t) = \int_0^{L-t} \mu x_2(\tau)e^{-\mu \tau}d\tau + \Delta.
\] (9)

In light of the solution to Eq. (7), it is not hard to find that \( x_2(t) = \Delta \cdot (y(t) + 1) \). This finishes the proof. □

The condition that the void distance does not change during a transmission delay appears to exclude the case of bi-directional traffic especially when the delay is significant. In addition, following the solution procedure for \( V(X(t)) \), it is feasible to get the variance of propagation distance in the existence of transmission delay. We leave this to interested readers.

6. Conclusion and further discussions

Inter-vehicle communication has been under study for years as a promising technology to support a decentralized traffic information system. Effective and reliable information propagation through inter-vehicle communication plays an important role in the design of decentralized traffic information systems. We have developed closed form formulas for the expected value and variance of propagation distance starting with a void distance. We also consider the case with transmission delay. With these formulas, the relationship between propagation distance, equipped vehicle density, and transmission range becomes transparent. Furthermore, the formulas for the expected propagation distance and the expected number of relays when the
starting void distance equals to zero rectify the results in Cheng and Robertazzi (1989) in its one-dimensional case.

From the analytical formulas, we find that the number of vehicles $\mu L$ within each transmission range plays a critical role in successful propagation of information.

An impressive observation is that information propagation distance follows closely a Gamma distribution at practical $\mu L$ values. Our analytical results are in general consistent with those such as Figure 3.5, 3.8 as well as 4.3 from simulations in Yang (2003).

These results may add to the essential tool kit critical to system design and reliability analysis. Furthermore, it may provide a starting point to approach the problem of information propagation on a network, which is beyond the discussion in this paper.

Note that this study has an underlying assumption that no information packet is lost due to transmission conflict and that the mobile ad hoc network is capable of handling the generated information traffic. As a matter of fact, it depends on communication protocols and routing algorithms to some extent. However, we have confidence that our results apply generally under reasonable and practical protocols and algorithms.

### 6.1. Gamma-like behavior on heterogeneous traffic

An important future direction might be the study on information propagation over a heterogeneous traffic. A heterogeneous traffic consists of segments of varying density. Our preliminary study indicates a smaller expected propagation distance and a larger variance than those from using the average traffic density when $\mu L$ is relatively large (Note: $\mu$ is the average density). However, a Gamma-like behavior still appears obvious according to our simulations.

Fig. 8 shows a Gamma approximation to the propagation distance distribution over a heterogeneous traffic stream. In our simulation, the traffic stream consists of 11 different segments, each having a density between 0 and 12.0 at equal probability: $0, 0.6, 1.2, \ldots, 11.4, 12.0$ vehicles per unit length. Be aware that each segment is a homogeneous traffic section. In each succeeding segment, each density has the same presence probability. The length of each segment has an exponential distribution with a mean length set to be 0.12 (or, $0.3L$). We find that a Gamma distribution fits the observed cumulative frequency distribution well.

Study on information propagation over various heterogeneous traffic demands great effort. We leave this to future research.

### 6.2. Other applications

The relay process identified in this paper has many applications. In the following, we present some examples.

**Example 1.** Consider a treasure collection process along a street. A beggar searches for lost treasure on the street. His vision scope is $L$. If he finds any treasure within this scope, he goes and picks it up. After that he

![Fig. 8. Gamma-like behavior with heterogeneous traffic at $\mu = 6.0$ and $L = 0.4$.](image-url)
searches in the same direction again. If there is no treasure within his sight, he just returns. If presence of
treasure on the street follows an independent homogeneous Poisson process, the covered distance follows the
information relay process.

**Example 2.** Suppose an actuated signal is implemented at an intersection in one major direction. Whenever a
vehicle passes the intersection, the green light in that direction is extended by an additional length \( L \). If no
vehicle arrives with the extended green duration \( L \), the green light terminates. If vehicle arrivals follow an inde-
pendent homogeneous Poisson process, the time extension represents an information relay process.

**Example 3.** In the case of radio transmissions along a line of interest, if presence of transmission terminals
follows an independent homogeneous Poisson process and if the transmission range is \( L \) just as in the one-
dimensional case studied in Cheng and Robertazzi (1989), the transmission distance is a relay process.

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**Appendix A**

**A.1. Gamma distribution**

Gamma distribution has the following probability density function:

\[
f(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta},
\]

where \( x > 0, k > 0, \theta > 0 \), and \( \Gamma(k) \) is Gamma function defined as follows:

\[
\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt.
\]

Note that the Gamma distribution above has a mean \( k\theta \) and a variance \( k\theta^2 \).

When \( k \to 1 \), Gamma distribution approaches exponential. Exponential distribution is given in the follow-
ing form.

\[
f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}
\]

Its mean and variance both become \( \theta \).

**A.2. Proof of Theorem 2**

Taking derivative to \( l \), Eq. (3) gives the following result:

\[
\dot{x}(l) = -\mu(l + x(l - l)) e^{-\mu(l - l)}.
\]

(10)

Obviously, we need to change \( x(l) \) and \( x(L - l) \) into the same format in order to solve the differential equation.

From Eq. (10), letting \( t = L - l \) gives

\[
\dot{x}(L - t) = -\mu(L - t + x(t)) e^{-\mu t}.
\]

(11)

Slightly rearranging the terms gives

\[
\mu x(t) = -\dot{x}(L - t)e^{\mu t} - \mu(L - t).
\]

(12)
Taking derivative to $t$ at both sides leads to

$$\mu \dot{x}(t) = \ddot{x}(L-t)e^{\mu t} - \mu \dot{x}(L-t)e^{\mu t} + \mu. \quad (13)$$

Substituting $\ddot{x}(L)$ in Eq. (10) into Eq. (13) gives the following:

$$-\mu^2(t + x(L-t))e^{-\mu(L-t)} = \ddot{x}(L-t)e^{\mu t} - \mu \dot{x}(L-t)e^{\mu t} + \mu.$$ 

Rearranging the terms, one has

$$\ddot{x}(L-t) - \mu \dot{x}(L-t) + x(L-t)\mu^2 e^{-\mu t} = -\mu e^{\mu t} - \mu^2 te^{-\mu t}.$$ 

Let $y = L - t$ and then $t = y$ (we call switching $L - t$ and $t$ in the future for simplicity). We have

$$\ddot{x}(t) - \dot{x}(t)\mu + \mu^2 e^{-\mu t} x(t) = -\mu e^{\mu t} - \mu^2(L-t)e^{-\mu t}. \quad (14)$$

Note that the coefficients of the left hand side are all constants. Differential equation (14) represents a typical ordinary second-order differential equation. Its solution follows the superposition principle, and is the sum of two components: general solution that makes the left hand equal to zero (denoted by $x_d(t)$) and special solutions that make the left hand equal to each of the terms at the right hand side respectively. Solution details to this type of ordinary differential equations are available in many textbooks (see, for example, Andrews, 1982, Chapter 4).

The auxiliary equation of Eq. (14) is as follows:

$$r^2 - \mu r + \mu^2 e^{-\mu t} = 0.$$ 

Following the standard solution process, we can easily obtain the solution to Eq. (14). The coefficients in the solution can be determined by substituting the solution back into Eq. (10) or Eq. (3). Details are omitted for the interest of space. □

A.3. Proof of Theorem 3

We will present our formula based on Eves Rule (see Bain and Engelhardt, 1991, Chapter 5, Theorem 5.4.3) for variance.

$$V(X) = V[E[X|Y]] + E[V(X|Y)],$$

where we denote by $V(\cdot)$ the variance. Eves Rule in our case is expressed as follows:

$$V(X(t)) = V(E[X(t)|\tau]) + E[V(X(t)|\tau)], \quad (15)$$

where $\tau$ is the location of the furthest equipped vehicle within range. Using the fact that $E[E[X(t)|\tau]] = x(t)$, first term of the right hand side is calculated as follows:

$$V(E[X(t)|\tau]) = \int_0^{L-t} \mu e^{-\mu t} (L - \tau + X(\tau) - x(t))^2 d\tau + e^{-\mu(L-t)}(0 - x(t))^2$$

$$= \int_0^{L-t} \mu e^{-\mu t} \left( L - \tau + \tau - L - \frac{1}{\mu} e^{\mu t} + \frac{1}{\mu} e^{\mu t} - x(t) \right)^2 d\tau$$

$$+ e^{-\mu(L-t)}(0 - x(t))^2$$

(substituting $x(t$) = $\frac{1}{\mu} e^{\mu(L-t)} - \frac{1}{\mu^2} t e^{\mu t} + \frac{2}{\mu} (t - L)e^{\mu t} - \frac{1}{\mu^2} e^{2\mu t} + \frac{1}{\mu^2} e^{\mu(L+t)} - (t - L)^2$.

Note that $V(X(t)|\tau) = V(L - \tau + X(\tau)|\tau) = V(X(\tau)|\tau)$. To be simple, we use $V(X(\tau))$ for $V(X(\tau)|\tau)$. The second term of the right hand side is then given as follows:

$$E[V(X(t)|\tau)] = \int_0^{L-t} \mu e^{-\mu t} V(L - \tau + X(\tau)|\tau)d\tau = \int_0^{L-t} \mu e^{-\mu t} V(X(\tau))d\tau.$$
Therefore, we have
\[
V(X(t)) = V(E[X(t)|\tau]) + \int_0^{L-t} \mu e^{-\mu t} V(X(\tau)) \, dt.
\] (17)

Taking derivative to \( t \) at both sides leads to
\[
\dot{V}(X(t)) = \dot{V}(E[X(t)|\tau]) - \mu e^{-\mu(L-t)} V(X(L-t)),
\] (18)

where
\[
\dot{V}(E[X(t)|\tau]) = -\frac{1}{\mu} e^{\mu(t-L)} + \frac{2}{\mu} e^{\mu t} + 2(t-L)e^{\mu t} - \frac{2}{\mu} e^{2\mu t} + \frac{1}{\mu} e^{\mu(L-t)} - 2t + 2L.
\]

Switching \( t \) and \( L - t \) in Eq. (18) gives the following:
\[
\dot{V}(X(L-t)) = -\mu e^{-\mu t} V(X(t)) - \frac{1}{\mu} e^{\mu t} + \frac{2}{\mu} e^{\mu(L-t)} - 2e^{2\mu t} + \frac{1}{\mu} e^{\mu(2L-t)} + 2t.
\] (19)

Taking derivative to \( t \) at both sides of (19) gives
\[
-\dot{V}(X(L-t)) = \mu^2 e^{-\mu t} V(X(t)) - \mu e^{-\mu t} \dot{V}(X(t)) - e^{\mu t} - 2e^{2\mu t} + 2\mu e^{\mu t} e^{\mu(L-t)}
- 2e^{\mu(L-t)} + 4e^{2\mu(L-t)} - e^{\mu(2L-t)} + 2.
\] (20)

Furthermore, Eq. (19) gives
\[
\mu^2 e^{-\mu t} V(X(t)) = -\mu \dot{V}(X(L-t)) - e^{\mu t} + 2e^{2\mu t} - 2\mu e^{\mu t} - 2e^{2\mu t} + e^{\mu(2L-t)} + 2\mu t.
\] (21)

In addition, Eq. (18) gives
\[
-\mu e^{\mu t} \dot{V}(X(t)) = e^{\mu(L-2t)} - 2 - 2\mu(t-L) + 2e^{\mu t} - e^{2\mu t} + 2\mu(t-L)e^{-\mu t} + \mu^2 e^{-\mu L} V(X(L-t)).
\] (22)

Substituting (21) and (22) back into (20), we have the following ordinary second-order differential equation:
\[
-\dot{V}(X(L-t)) = -2e^{\mu(t-L)} + 2e^{2\mu(t-L)} - \mu \dot{V}(X(L-t)) + e^{\mu(L-2t)} + 2\mu L - e^{\mu L}
+ 2\mu(t-L)e^{-\mu t} + \mu^2 e^{-\mu L} V(X(L-t)).
\]

Switching between \( t \) and \( L - t \) gives
\[
-\dot{V}(X(t)) = -2e^{\mu t} + 2e^{2\mu t} - \mu \dot{V}(X(t)) + e^{\mu(2t-L)} + 2\mu L - e^{\mu L} - 2\mu e^{-\mu(L-t)} + \mu^2 e^{-\mu L} V(X(t)).
\]

The above equation has a standard format as follows:
\[
\ddot{V}(X(t)) - \mu \dot{V}(X(t)) + \mu^2 e^{-\mu L} \dot{V}(X(t)) = 2e^{\mu t} - 2e^{2\mu t} - e^{\mu(2t-L)} - 2\mu L + e^{\mu L} + 2\mu e^{-\mu(L-t)}.
\] (23)

Following the standard solution to the above ordinary differential equation finishes the proof. □

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