Equations for the problem of a thin finite plate on a Pasternak foundation

Part 1: Closed Integrals
Equations for the problem of a thin finite plate
on a Pasternak foundation

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Report: W-DWW-98-010
Date: January 1998

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Equations for the problem of a thin finite plate
on a Pasternak foundation

Introduction

In concrete pavement design the pavement is often regarded as or simulated by a thin plate resting on an elastic foundation. In the past the Winkler foundation was the commonly used model for the description of the foundation response. Nowadays the Pasternak foundation, a two parameter model, is a better simulation for the response. It allows for shear force transfer through the foundation which is not possible with the Winkler foundation (uncoupled vertical spring system). For this type of foundations in combination of a thin plate simulation for the concrete layers on top the big advantage is that stresses and strains can be calculated at the boundary of a finite plate. In this document the equations are given for two adherent semi-infinite plates of which the left one is loaded by a homogenous stress distribution within a rectangle with sides $2a$ and $2b$ and its centre in $x=0$ and $y=0$. The joint between the two plates is located at $x=d$ normal to the $x$ axis. The differential equation is given in equation 1.

$$N \Delta \Delta (W) - G \Delta (W) + kW = p$$  \hspace{1cm} (1)

$W$: Deflection at $x,y$ \hspace{1cm} [m]
$k$: Modulus of subgrade reaction \hspace{1cm} [N/m$^3$]
$G$: Shear modulus for foundation \hspace{1cm} [N/m]
$N$: Bending stiffness parameter \hspace{1cm} [Nm]

For a rectangular load with sides $2a$ and $2b$ and a homogenous stress distribution $p$ the solution for an infinite plate is given by equation 2.

$$\frac{4pN}{\pi^2} \int_0^\infty \int_0^\infty \frac{\cos(t\frac{x}{L})\sin(t\frac{a}{L})\cos(s\frac{y}{L})\sin(s\frac{b}{L})}{ts(t^4 + 2t^2s^2 + 2gt^2 + 2gs^2 + s^4 + 1)} \, dt \, ds$$

If the denominator is split into three terms (with respect to $t$) as given in Annex I, this double integral can be simplified into a single integral of the form:

$$\frac{2p}{\pi k} \int_0^\infty \left[ \frac{\cos(y\frac{s}{L})\sin(b\frac{s}{L})}{s} \times (F_1(x,s) + F_2(x,s) + F_3(x,s)) \right] \, ds$$  \hspace{1cm} (3)

The main content of this document deals with the problem of two adherent semi-infinite plates on a Pasternak foundation. The two adherent plates make no physical contact (free plate condition). Therefore no transfer of moment is possible at the joint. Transfer of shear forces will take place through the shear layer in the foundation. However, in Annex 2 an overview is given of the more general problem in which transfer of moment is also possible at the joint. This situation might often occur in practice if a pre-sawn crack in the plate is not completely cracked leading to a free joint condition or if stiff dowels at the joint are capable in transferring an amount of moment.
Thin Plate on a Pasternak Foundation

Solutions of the differential equation with and without loading

Outside the rectangular loaded area (sides a and b) and for the first quadrant 
\( x > a \); \( y > 0 \) the deflection \( W_{\text{inf}} \) is given by equation 4 in which \( F_{\text{inf}} \) is defined by

\[
\frac{2p}{\pi k} \int_0^s \cos \left( \frac{y^s}{L} \right) \sin \left( \frac{b^s}{L} \right) \frac{1}{s} \times F_{\text{inf};x=a} \, ds
\]

\( (4) \)

\[
F_{\text{inf};x=a} = -B_s \cdot e^{-z_1 \frac{x}{L}} \cdot \sinh \left( z_1 \frac{a}{L} \right) - C_s \cdot e^{-z_2 \frac{x}{L}} \cdot \sinh \left( z_2 \frac{a}{L} \right)
\]

\( (5) \)

Replacing \( F_{\text{inf},x=a} \) in equation 4 by \( F_{\text{left}} \) (equation 6) represents the solutions of the homogenous differential equation which are valid for \( -\infty < x < d \) :: left side plate

\[
F_{\text{left}} = A \cdot e^{-z_1 \frac{x}{L}} \cdot \sinh \left( z_1 \frac{a}{L} \right) + B \cdot e^{-z_2 \frac{x}{L}} \cdot \sinh \left( z_2 \frac{a}{L} \right)
\]

\( (6) \)

in equation 4 by \( F_{\text{right}} \) (equation 7) represents the solutions of the homogenous differential equation which are valid for \( d < x < +\infty \) :: plate right to the boundary.

\[
F_{\text{right}} = C \cdot e^{-z_1 \frac{x}{L}} \cdot \sinh \left( z_1 \frac{a}{L} \right) + D \cdot e^{-z_2 \frac{x}{L}} \cdot \sinh \left( z_2 \frac{a}{L} \right)
\]

\( (7) \)

The complete deflection \( W (x>a) \) on the finite plate left to the boundary at \( x=d \) is now represented by equation 8 and for the plate right to the boundary by equation 9.

\[
W = \frac{2p}{\pi k} \int_0^s \cos \left( \frac{y^s}{L} \right) \sin \left( \frac{b^s}{L} \right) \frac{1}{s} \times [F_{\text{inf};x=a} + F_{\text{left}}] \, ds
\]

\( (8) \)

\[
W = \frac{2p}{\pi k} \int_0^s \cos \left( \frac{y^s}{L} \right) \sin \left( \frac{b^s}{L} \right) \frac{1}{s} \times F_{\text{right}} \, ds
\]

\( (9) \)

The following abbreviations are introduced:

\[
- B_s \cdot \sinh \left( z_1 \frac{a}{L} \right) = B_s^* \cdot e^{-z_1 \frac{d}{L}} \quad ; \quad - C_s \cdot \sinh \left( z_2 \frac{a}{L} \right) = C_s^* \cdot e^{-z_2 \frac{d}{L}}
\]

\( (10) \)
\[ A \cdot \sinh\left(\frac{z_1 a}{L}\right) = A^* \cdot e^{-\frac{z_1 d}{L}}; \quad B \cdot \sinh\left(\frac{z_2 a}{L}\right) = B^* \cdot e^{-\frac{z_2 d}{L}} \]  

(11)

\[ C \cdot \sinh\left(\frac{z_1 a}{L}\right) = C^* \cdot e^{\frac{z_1 d}{L}}; \quad D \cdot \sinh\left(\frac{z_2 a}{L}\right) = D^* \cdot e^{\frac{z_2 d}{L}} \]  

(12)

This introduction simplifies the boundary equations at x=d
Equations 5, 6 and 7 are now written as:

\[ F_{\text{out}} = B_{s}^* \cdot e^{-\frac{z_1 x}{L}} \cdot e^{-\frac{z_1 d}{L}} + C_{s}^* \cdot e^{-\frac{z_2 x}{L}} \cdot e^{\frac{z_2 d}{L}} \]  

(13)

\[ F_{\text{left}} = A^* \cdot e^{\frac{z_1 x}{L}} \cdot e^{-\frac{z_1 d}{L}} + B^* \cdot e^{\frac{z_2 x}{L}} \cdot e^{-\frac{z_2 d}{L}} \]  

(14)

\[ F_{\text{right}} = C^* \cdot e^{-\frac{z_1 x}{L}} \cdot e^{\frac{z_1 d}{L}} + D^* \cdot e^{-\frac{z_2 x}{L}} \cdot e^{\frac{z_2 d}{L}} \]  

(15)

Boundary requirements based on the equilibrium for a free plate condition.

The moments normal to the boundary at x=d have to be zero. This moment is:

\[ M_{x; x=1 d} = -N \left( \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right) W = 0 \]  

(16)

For the plate right to the boundary this leads to equation 17:

\[ \left( z_1^2 - \mu \cdot s^2 \right) C^* + \left( z_2^2 - \mu \cdot s^2 \right) D^* = 0 \]  

(17)

Introducing the abbreviations \( u_1 \) and \( u_2 \) according to equation 18, equation 17 can be rewritten as equation 19:

\[ u_1 = z_1^2 - \mu \cdot s^2; \quad u_2 = z_2^2 - \mu \cdot s^2 \]  

(18)

\[ u_1 \cdot C^* + u_2 \cdot D^* = 0 \]  

(19)

For the plate left to the boundary equation 20 is obtained for the requirement.

\[ u_1 \cdot B_s^* + u_2 \cdot C_s^* + u_1 \cdot A^* + u_2 \cdot B^* = 0 \]  

(20)
At the boundary shear force transfer will take place through the shear layer in the foundation. Here the requirement is continuity of the shear forces \( T \) (including the twisting moment) in the plate and the shear force \( S \) in the shear layer. The total shear force is given by equation 21

\[
[T + S]_{x=d} = -N_s \left[ \frac{\partial^3}{\partial x^3} + (2 - \mu) \frac{\partial^3}{\partial x \partial y^2} - \frac{2g}{L^2} \frac{\partial}{\partial x} \right] W
\]

This boundary requirements leads to equation 22

\[
-z_1^3 \cdot B_s^* - z_2^3 \cdot C_s^* + z_1^3 \cdot A^* + z_2^3 \cdot B^*
- (2 - \mu) s^2 \left[ -z_1 \cdot B_s^* - z_2 \cdot C_s^* + z_1 \cdot A^* + z_2 \cdot B^* \right]
- 2g \left[ -z_1 \cdot B_s^* - z_2 \cdot C_s^* + z_1 \cdot A^* + z_2 \cdot B^* \right] =
- z_3^3 \cdot C^* - z_2^3 \cdot D^* - (2 - \mu) s^2 \left[ -z_1 \cdot C^* - z_2 \cdot D^* \right]
- 2g \left[ -z_1 \cdot C^* - z_2 \cdot D^* \right]
\]

Rewriting of equation 22 leads to:

\[
-z_1 \left[ z_1^2 - (2 - \mu) s^2 - 2g \right] \cdot B_s^* - z_2 \left[ z_2^2 - (2 - \mu) s^2 - 2g \right] \cdot C_s^* + z_1 \left[ z_1^2 - (2 - \mu) s^2 - 2g \right] \cdot \cdot A^* + z_2 \left[ z_2^2 - (2 - \mu) s^2 - 2g \right] \cdot B^* =
- z_1 \left[ z_1^2 - (2 - \mu) s^2 - 2g \right] \cdot C^* - z_2 \left[ z_2^2 - (2 - \mu) s^2 - 2g \right] \cdot D^*
\]

Because \( z_1^2 \) and \( z_2^2 \) are defined as:

\[
z_1^2 = s^2 + g + \sqrt{g^2 - 1}; \quad z_2^2 = s^2 + g - \sqrt{g^2 - 1}
\]

the following simplification can be used:

\[
z_1^2 - (2 - \mu) s^2 - 2g = -s^2 - g + \sqrt{g^2 - 1} + \mu s^2 = -z_2^2 + \mu s^2 = -u_2
\]

\[
z_2^2 - (2 - \mu) s^2 - 2g = -s^2 - g - \sqrt{g^2 - 1} + \mu s^2 = -z_1^2 + \mu s^2 = -u_1
\]

Equation 22 can now be rewritten as:

\[
z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^* - z_1 u_2 \cdot A^* - z_2 u_1 \cdot B^* = z_1 u_2 \cdot C^* + z_2 u_1 \cdot D^*
\]

Elimination of \( D^* \) from equations 19, 20 and 26 leads to:

\[
z_1 u_2^2 \cdot B_s^* + z_2 u_1 u_2 \cdot C_s^* - z_1 u_2^2 \cdot A^* - z_2 u_1 u_2 \cdot B^* = (z_1 u_2^2 - z_2 u_1^2) \cdot C^*
\]
Elimination of $B^*$ from equations 20 and 27 gives:

\[(z_2 u_1^2 + z_1 u_2^2) \cdot B_s^* + 2z_2 u_1 u_2 \cdot C_s^* + (z_2 u_1^2 - z_1 u_2^2) \cdot A^* = (z_1 u_2^2 - z_2 u_1^2) \cdot C^*\]  \hspace{1cm} (28)

and also $A^*$ and $C^*$ from equations 19, 20 and 26 gives:

\[2z_1 u_1 u_2 \cdot B_s^* + (z_1 u_2^2 + z_2 u_1^2) \cdot C_s^* + (z_1 u_2^2 - z_2 u_1^2) \cdot B^* = (z_2 u_1^2 - z_1 u_2^2) \cdot D^*\]  \hspace{1cm} (29)

\[\xi = z_2 u_1^2 - z_1 u_2^2\]  \hspace{1cm} (30)

Using equation 30 the boundary equations for the equilibrium condition are:

\[A^* = -C^* - \frac{(z_2 u_1^2 + z_1 u_2^2)}{\xi} \cdot B_s^* - \frac{2z_2 u_1 u_2}{\xi} \cdot C_s^*\]  \hspace{1cm} (31)

\[B^* = -D^* + \frac{2z_1 u_1 u_2}{\xi} \cdot B_s^* + \frac{(z_1 u_2^2 + z_2 u_1^2)}{\xi} \cdot C_s^*\]  \hspace{1cm} (32)

\[u_1 \cdot C^* + u_2 \cdot D^* = 0\]  \hspace{1cm} (33)

**Fourth Boundary Equation**  \hspace{1cm} **Case I: Constant ratio in deflections**

One valid boundary requirement is given by equation 34 implying a difference in deflections between the plate left to the boundary and the plate right to the boundary:

\[W_{right} = \gamma \cdot W_{left}; \hspace{0.5cm} 0 < \gamma < 1\]  \hspace{1cm} (34)

\[\gamma \cdot [B_s^* + C_s^* + A^* + B^*] = C^* + D^*\]  \hspace{1cm} (35)

Solving equations 31, 32, 33 and 34 the coefficients $A^*$, $B^*$, $C^*$ and $D^*$ are:

\[A^* = \frac{1}{1 + \gamma} \cdot \frac{1}{\xi} \left\{ \left( (\gamma - 1) z_1 u_2^2 - (\gamma + 1) z_2 u_1^2 \right) \cdot B_s^* + 2z_2 u_1 u_2 \cdot C_s^* \right\}\]  \hspace{1cm} (36)

\[B^* = \frac{1}{1 + \gamma} \cdot \frac{1}{\xi} \left\{ 2z_1 u_1 u_2 \cdot B_s^* + \left\{ (\gamma + 1) z_1 u_2^2 - (\gamma - 1) z_2 u_1^2 \right\} \cdot C_s^* \right\}\]  \hspace{1cm} (37)

\[C^* = -\frac{2 \gamma}{1 + \gamma} \cdot \frac{u_2}{\xi} \left[ z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^* \right]\]  \hspace{1cm} (38)

\[D^* = +\frac{2 \gamma}{1 + \gamma} \cdot \frac{u_1}{\xi} \left[ z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^* \right]\]  \hspace{1cm} (39)
At this stage we return to the original coefficients using:

\[
B_s^* = -e^{-z_1^d L}B_s \cdot sinh\left(z_1^d L\right); \quad C_s^* = -e^{-z_2^d L}C_s \cdot sinh\left(z_2^d L\right)
\]  \hfill (40)

\[
A \cdot sinh\left(z_1^a L\right) = A^* \cdot e^{-z_1^d L}; \quad B \cdot sinh\left(z_2^a L\right) = B^* \cdot e^{-z_2^d L}
\]  \hfill (41)

\[
C \cdot sinh\left(z_1^a L\right) = C^* \cdot e^{z_1^d L}; \quad D \cdot sinh\left(z_2^a L\right) = D^* \cdot e^{z_2^d L}
\]  \hfill (42)

\[
A_s = \frac{1}{z_1^2 z_2^2}; \quad B_s = +\frac{1}{z_1^2} - \frac{1}{z_1^2 - z_2^2}; \quad C_s = -\frac{1}{z_2^2}, \frac{1}{z_1^2 - z_2^2}
\]  \hfill (43)

\[
z_1^2 = s^2 + g + \sqrt{g^2 - 1}; \quad z_2^2 = s^2 + g - \sqrt{g^2 - 1}
\]  \hfill (44)

\[
z_1^2 z_2^2 = s^4 + 2gs^2 + 1; \quad z_1^2 - z_2^2 = 2\sqrt{g^2 - 1}
\]  \hfill (45)

\[
u_1 = z_1^2 - \mu s^2; \quad \nu_2 = z_2^2 - \mu s^2; \quad \xi = z_2 \nu_1^2 - z_1 \nu_2^2
\]  \hfill (46)

The deflection \(W\) for \(0 < x < a\) (including the loaded area of the first quadrant) can be calculated with equation 8 using equation 47 (\(W_{inf,0 < x < a}\)) for \(W_{inf,x > a}\) (equation 5).

\[
F_{w,0 < x < a} = A_s + B_s \cdot e^{-z_1^d L} \cdot cosh\left(z_1^d L\right) + C_s \cdot e^{-z_2^d L} \cdot cosh\left(z_2^d L\right)
\]  \hfill (47)

The formulas for \(W_{inf}\) in the other quadrants are obtained by changing signs in the exponents which will ensure that the functions will go to zero for \(x\) going to infinity.

Review of the deflection calculation formulas by:

\[
W = \frac{2p}{\pi k} \int_0^x \left[\frac{cos(y^S)}{L} \cdot \frac{sin(b^S)}{s} \cdot \frac{F_1 + F_2}{s}ight] ds
\]  \hfill (48)

\[0 < x < a : \quad F_1 = \text{equation 47} \quad ; \quad F_2 = \text{equation 6}\]

\[a < x < d : \quad F_1 = \text{equation 5} \quad ; \quad F_2 = \text{equation 6}\]

\[d < x < \infty : \quad F_1 = 0 \quad ; \quad F_2 = \text{equation 7}\]

Remarks: In numerical calculations the term \(z_1^2 - z_2^2\) must be replaced by its constant value \(2\sqrt{g^2 - 1}\), that part of the integral which contains the coefficient \(A_s\) must be solved analytical (see Annex) and for \(s = 0\) the limit of \(\sin(b.s/L)/s = b/L\) should be taken.
Fourth Boundary Equation

Case II: Difference in deflections is related to shear force

Another possible fourth boundary condition is given by equation 49

\[
\frac{N}{L^3} \cdot (1 - \alpha) \cdot [W_{\text{left}} - W_{\text{right}}] = -N \cdot \alpha \cdot \left[ \frac{\partial^3}{\partial x^3} + (2 - \mu) \frac{\partial^3}{\partial x \partial y^2} - \frac{2g}{L^2} \frac{\partial}{\partial x} \right] W \tag{49}
\]

This boundary condition is used in linear elastic multi-layer programs for describing the interface condition between layers. For \(\alpha = 0\) in the linear elastic multi-layer programs the condition of full friction is obtained, which is equal to the condition of continuity in deflections as postulated by Kerr. For \(\alpha = 1\) the condition of full slip is obtained, which reflects the situation of a complete free boundary (no shear transfer).

Using the same abbreviations as before the following fourth boundary equation is obtained:

\[
A^* + B^* = -B^* - C_s^* + C^* \left( 1 + \frac{\alpha}{1 - \alpha} \cdot z_1 u_2 \right) + D^* \left( 1 + \frac{\alpha}{1 - \alpha} \cdot z_2 u_1 \right) \tag{50}
\]

Subtracting of equations 31 and 32 leads to:

\[
\frac{2(u_1 - u_2)}{\xi} [z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^*] = -\left[ 2 + \frac{\alpha z_1 u_2}{1 - \alpha} \right] C^* + \left[ 2 + \frac{\alpha z_2 u_1}{1 - \alpha} \right] D^* = 0 \tag{51}
\]

If the same procedure is performed with equations 31, 32 and 35 (case I) the following equation is obtained:

\[
\frac{2(u_1 - u_2)}{\xi} [z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^*] - \left[ 1 + \frac{1}{\gamma} \right] C^* - \left[ 1 + \frac{1}{\gamma} \right] D^* = 0 \tag{52}
\]

Due to the differentiation the equations seems not comparable. The coefficients \(C^*\) and \(D^*\) for case II are given by equations 53 and 54.

\[
C^* = -\frac{1}{1 + \frac{\alpha}{\xi} \cdot \frac{u_2}{2(u_1 - u_2)(1 - \alpha)}} \cdot [z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^*] \tag{53}
\]

\[
D^* = +\frac{1}{1 + \frac{\alpha}{\xi} \cdot \frac{u_1}{2(u_1 - u_2)(1 - \alpha)}} \cdot [z_1 u_2 \cdot B_s^* + z_2 u_1 \cdot C_s^*] \tag{54}
\]

The two extreme case are equal (\(\alpha = 0 \Rightarrow \gamma = 1\) and \(\alpha = 1 \Rightarrow \gamma = 0\) \(\Rightarrow C^* = D^* = 0\)). The first extreme represents complete friction (no difference in deflections). The second extreme case represents complete slip (no shear force transfer at the boundary).

Note: The term \(u_1 - u_2\) is equal to \(z_1^2 - z_2^2 = 2\sqrt{[g^2 - 1]}\).
Using the abbreviation given in equation 55, the coefficients $A^*$ and $B^*$ are represented for this condition by equations 56 and 57.

\[
\zeta = \frac{\alpha \xi}{2 (1 - \alpha) (u_1 - u_2)} = \frac{\alpha \xi}{4 (1 - \alpha) \sqrt{g^2 - 1}} \quad (55)
\]

\[
A^* = -\frac{1}{1 + \zeta} \frac{1}{\xi} \left[ \left( \zeta z_1 u_2^2 + (1 + \zeta) z_2 u_1^2 \right) B_s^* + \left( 1 + 2 \zeta \right) z_2 u_1 u_2 C_s^* \right] \quad (56)
\]

\[
B^* = \frac{1}{1 + \zeta} \frac{1}{\xi} \left[ \left( 1 + 2 \zeta \right) z_1 u_1 u_2 B_s^* + \left( 1 + \zeta \right) z_1 u_2^2 + \zeta z_2 u_1^2 \right] C_s^* \quad (57)
\]

Although not directly visible the equations 36 to 39 (case I) and equations 53, 54, 55 and 56 are in principle the same if the parameter $\xi$ is replaced by $(1-\gamma)/(2\gamma)$. Therefore the two cases are identical with respect to the equations. However, the parameter $\gamma$ is a constant while $\xi$ depends on the integration parameter $s$.

**Load transfer coefficient (LTC)**

The LTC is defined as the ratio of the shear force in the plate (and the foundation) at the joint and the shear force at that point if no joint is present (infinite plate).

\[
LTC = \frac{\tau_{x=d}}{\tau_{inf.}} \quad (58)
\]

If the (constant) ratio of the deflections at both sides of the joint is taken as the fourth boundary condition (equation 34), it can be easily shown that there is a direct relationship between this ratio and the LTC parameter. The shear force at the joint is represented by the two terms ($C^*, D^*$) at the right side of equation 26. The shear force in an infinite plate is represented by the first two terms ($B_s^*, C_s^*$) at the left side of this equation. Replacing $C^*$ and $D^*$ by $B_s^*$ and $C_s^*$ (using equations 36 and 37) leads directly to the relationship:

\[
LTC = \frac{2 \gamma}{1 + \gamma} \quad (59)
\]

This relationship was first established by F. van Cauwelaert for a plate on a Winkler foundation ($g=0$) and for a beam on a Pasternak foundation but as shown here it is also valid for the general case of $g > 0$. If instead of the ratio of the deflections, equation 49 is used as the fourth boundary condition (relationship between the difference in deflections and the shear force), the following expression for the LTC value will be obtained using the same procedure as in case I.

\[
LTC = \frac{1}{1 + \zeta} = \frac{1}{1 + \frac{\alpha (z_2 u_1^2 - z_1 u_2^2)}{4 (1 - \alpha) \sqrt{g^2 - 1}}} \quad (60)
\]
If \( \alpha = 0 \) (\( W_{\text{left}} = W_{\text{right}} \)) the LTC=1 and for \( \alpha = 1 \) the LTC=0 (T+S=0).
For this boundary condition and adopting the procedure for the calculation of the ratio of the deflections will lead to equation 61:

\[
\gamma = \frac{1}{1 + 2 \zeta} \tag{61}
\]

In return it looks like equation 61 can be used to express the LTC value in the deflection ratio \( \gamma \). This leads to:

\[
LTC = \frac{1}{1 + \zeta} = \frac{2 \gamma}{1 + \gamma} \tag{62}
\]

Thus it seems that the LTC value for this boundary condition (case II) is the same expression as the one deduced for case I.

However, the procedure consists out of: neglecting the integral and performing the calculations for the ratios (deflections and shear stresses) using only the integrand (the coefficients). This procedure is only allowed if the ratios of the integrands are constants as in case I (\( \gamma = \text{constant} \)). In case II the ratios of the integrands depend on the integration variable \( s \). Therefore, the correct ratios (LTC and \( \gamma \)) will depend on the coordinate \( y \) along the joint at \( x=d \) and must be calculated using the final expression (thus including the integral with respect to \( y \)).
A single semi-infinite plate

If no plate is present at the right side of the joint the (homogenous) differential for this side will be:

\[-G \Delta \{W\} + k.W = 0\]  \hspace{1cm} (63)

The solution of this equation is given by equation 64 which is valid for \(x \to \infty\) and is written in the same form as in the case of a thin plate on top of the foundation.

\[W_{right} = \frac{2P}{\pi k} \int_{0}^{s} \frac{\sin\left(\frac{sa}{L}\right) \cos\left(\frac{sy}{L}\right)}{s} \cdot C(s) \cdot e^{-\sqrt{s^2 + \frac{1}{2g} \cdot \frac{x-d}{L}}} \, ds\]  \hspace{1cm} (64)

Denoting the coefficient in the exponent by \(m/L\) according to:

\[\sqrt{\frac{s^2 + \frac{1}{2g}}{L}} = \frac{m_s}{L}\]  \hspace{1cm} (65)

the following two boundary equations are obtained for the moment and the shear force transfer:

\[u_1.B_s^* + u_2.C_s^* + u_1.A^* + u_2.B^* = 0\]  \hspace{1cm} (66)

\[z_1 u_2.B_s^* + z_2 u_1.C_s^* - z_1 u_2.A^* - z_2 u_1.B^* = 2 g m_s \cdot C^*\]  \hspace{1cm} (67)

The third boundary condition is given by:

\[\gamma \cdot (B_s^* + C_s^* + A^* + B^*) = C^*\]  \hspace{1cm} (68)

which will lead to equations 69 to 71 with \(Q=2\sqrt{(g^2-1)}\)

\[C^* = \frac{2 Q \gamma}{2 Q \gamma g m_s + \xi} \cdot \left( z_1 u_2.B_s^* + z_2 u_1.C_s^* \right)\]  \hspace{1cm} (69)

\[A^* = \frac{-2}{2 Q \gamma g m_s + \xi} \cdot \left[ + Q \gamma g m_s + \frac{z_1 u_2^2 + z_2 u_1^2}{2} \right] \cdot B_s^* + z_2 u_1 u_2.C_s^*\]  \hspace{1cm} (70)

\[B^* = \frac{+2}{2 Q \gamma g m_s + \xi} \cdot \left[ z_1 u_1 u_2.B^* + \left( - Q \gamma g m_s + \frac{z_1 u_2^2 + z_2 u_1^2}{2} \right) \cdot C_s^* \right]\]  \hspace{1cm} (71)
No shear force transfer

For completeness we will also give the three boundary requirements in case moment transfer will occur but no transfer of shear forces is possible. In that case the three equations with respect to the equilibrium of forces and moments are:

\[ z_1 u_2 \cdot B^*_s + z_2 u_1 \cdot C^* = z_1 u_2 \cdot A^* - z_2 u_1 \cdot B^* = 0 \]  \hspace{1cm} (72)

\[ u_1 \cdot B^*_s + u_2 \cdot C^* + u_1 \cdot A^* + u_2 \cdot B^* = u_1 \cdot C^* + u_2 \cdot D^* \]  \hspace{1cm} (73)

\[ z_1 u_2 \cdot C^* + z_2 u_1 \cdot D^* = 0 \]  \hspace{1cm} (74)

Summing up \( u_1 \) times equation 72 and \( z_1 u_2 \) times equation 73 and using equation 74 leads to equation 75. Taking \( u_2 \) times equation 72 and \( z_1 u_1 \) times equation 74 will lead to equation 76.

\[ B^* = + D^* + \frac{2 z_1 u_1 u_2}{\xi} \cdot B^*_s + \frac{(z_1 u_2^2 + z_2 u_1^2)}{\xi} \cdot C^* \]  \hspace{1cm} (75)

\[ A^* = + C^* - \frac{(z_2 u_1^2 + z_1 u_2^2)}{\xi} \cdot B^*_s - \frac{2 z_2 u_1 u_2}{\xi} \cdot C^* \]  \hspace{1cm} (76)

Note the similarity with the equations 31 and 32 in case of no moment transfer.

Because of the high hypothetically character of these boundary requirements this specific situation is not worked further out. However, in Annex II the more general situation is dealt with (shear force and moment transfer) while that situation might occur in practice.
ANNEX 1 "Discontinuous Integrals"

An overview of definite integrals which can be useful in the closed integral solution procedure for differential equations in concrete pavement design.

\[
\int_0^{t^p} \left\{ \cos \left( \frac{\rho + \mu - v}{2} \pi \right) J_v(qt) + \sin \left( \frac{\rho + \mu - v}{2} \pi \right) Y_v(qt) \right\} \, dt = \frac{I_v(pz) \cdot K_v(qz)}{z^{p-1}} \quad \text{for } q > p \quad \text{and} \quad \frac{I_v(qz) \cdot K_v(pz)}{z^{p-1}} \quad \text{for } q > p
\]  

(77)

The stringent restrictions for this integral are: \( v + \mu < \rho < 4 \)

If the restriction \( v + \mu < \rho \) is not fulfilled the problem still can be solved in two ways. Normally the original denominator will contain the factor \((t^4 + z^4)\). To avoid this problem the denominator should be split into three terms according to equation 78.

\[
\frac{1}{t^4 + z^4} = A + \frac{B \cdot t^2}{t^2 + z_1^2} + \frac{C \cdot t^2}{t^2 + z_2^2}
\]  

(78)

If one knows that the solution is of the form of equation 77 but must contain an extra term \( z^n \) for one of the two conditions \((p < q \text{ or } q > p)\) one might try the so called Wronskians relation between the Bessel functions \( I_{\nu, \mu} \) and \( K_{\nu, \mu} \) (see equations 82 to 86).

The Bessel function \( Y_v \) will disappear for \( \rho + \mu - v = 2n \). \( \rightarrow \cos(n \pi) = (-1)^n \).

Special cases are:

- \( \rho = 0 ; \mu - v = 2n ; < 0 \rightarrow n \leq -1 \)
- \( \rho = 1 ; \mu - v = 2n - 1 ; < 1 \rightarrow n \leq 0 \)
- \( \rho = 2 ; \mu - v = 2n ; < 2 \rightarrow n \leq 0 \)

Handy relationships between Bessel functions and trigonometric functions are:

\[
\sin(pt) = \sqrt{\frac{pt \pi}{2}} \cdot J_{1/2}(pt) ; \quad \cos(pt) = \sqrt{\frac{pt \pi}{2}} \cdot J_{-1/2}(pt) \quad \text{for } p < 0
\]  

(79)

Also for hyperbolic functions:

\[
\sinh(pz) = \sqrt{\frac{2 
\pi}{p}} \cdot J_{1/2}(pz) ; \quad \cosh(pz) = \sqrt{\frac{2 
\pi}{p}} \cdot J_{1/2}(pz)
\]  

(80)

and for exponential functions:

\[
e^{-pz} = \sqrt{\frac{2 \pi}{pz}} \cdot [K_{1/2}(pz) = K_{1/2}(pz)]
\]  

(81)
The Wronskians $[F_1,F_2] \equiv \{ F_1 F'_2 - F_2 F'_1 \neq 0 \}$:

$$[J_\nu(z), J_{-\nu}(z)] = -\frac{2 \sin(\nu \pi)}{\pi z} \text{ for } \nu \neq \text{integer} \quad (82)$$

$$[I_\nu(z), I_{-\nu}(z)] = -\frac{2 \sin(\nu \pi)}{\pi z} \text{ for } \nu \neq \text{integer} \quad (83)$$

$$[J_\nu(z), Y_\nu(z)] = +\frac{2}{\pi z} \text{ for all } \nu \quad (84)$$

$$[I_\nu(z), K_\nu(z)] = -\frac{1}{z} \text{ for all } \nu \quad (85)$$

$$I_\nu(z) . K_{\nu+1}(z) + I_{\nu+1}(z) . K_\nu(z) = \frac{1}{z} \quad (86)$$

Special cases

$$\int_0^z \frac{\sin(pt) \cdot \cos(qt)}{t(t^2 + z^2)} \cdot dt = \frac{\pi}{2z^2} e^{-qt} \cdot \sinh(pz) \text{ for } q > p > 0 \quad (87)$$

$$\frac{\pi}{2z^2} [1 - e^{-pz} \cdot \cosh(qz)] \text{ for } p > q > 0$$

$$\int_0^z \frac{\sin(pt)}{t(t^2 + z^2)} \cdot dt = \frac{\pi}{2z^2} [1 - e^{-pz}] \text{ for } p > 0 \quad (88)$$

$$\int_0^z \frac{\sqrt{t} \cdot J_{1/2}(pt)}{t^2 + z^2} \cdot dt = \frac{1}{\sqrt{z}} \cdot K_{1/2}(pz) \quad (89)$$

$$\int_0^z \frac{J_{1/2}(pt)}{\sqrt{t}(t^2 + z^2)} \cdot dt = \frac{\pi}{2z\sqrt{z}} [L_{1/2}(pz) - L_{-1/2}(pz)] \quad (90)$$

In the last integral the function notation $L$ stands for Lommel's function.
ANNEX 2 "Pseudo Boundary Crossings"

The benefits of the closed integral solutions can easily been shown by regarding the requirements at a "pseudo" boundary. The pseudo boundary is a boundary at which the load on the plate or the formulation for the deflection changes but no changes occur in the response formulation for the behaviour of plate and foundation. Therefore there exists no real boundary while the deflection \( W \), the first derivative \( W'_1 \) (normal to the pseudo boundary), the moment \( M'_1 \) and the shear force \( T+S \) must be continuous at this crossing. Observing the equations involved this means that the function \( W \) itself and its first, second and third derivatives normal to the pseudo boundary must be continuous. We will show this for the solution of the loaded infinite plate at the crossing \( x=a \) (\( y>b \)). The integrand value (containing the dependency on \( x \)) at the right side of \( x=a \) is given by equation 91 and at the left side of \( x=a \) by equation 92:

\[
F_{w;x>a} = -B_s \cdot e^{-z_1^x_a} \cdot \sinh(z_1 \cdot \frac{a}{L}) - C_s \cdot e^{-z_2^x_a} \cdot \sinh(z_2 \cdot \frac{a}{L}) \tag{91}
\]

\[
F_{w;0<x<a} = A_s + B_s \cdot e^{-z_1^x_a} \cdot \cosh(z_1 \cdot \frac{x}{L}) + C_s \cdot e^{-z_2^x_a} \cdot \cosh(z_2 \cdot \frac{x}{L}) \tag{92}
\]

\[
A_s = \frac{1}{z_1^2 z_2^2} ; \quad B_s = \frac{1}{z_1^2} \cdot \frac{1}{z_1^2 - z_2^2} ; \quad C_s = -\frac{1}{z_2^2} \cdot \frac{1}{z_1^2 - z_2^2} \tag{93}
\]

The function value is continuously \([A_s+(B_s+C_s)/2 = -(B_s+C_s)/2]\). For the first and third derivatives the separate terms (containing the coefficients \( B_s \) and \( C_s \)) are continuously while for the second derivative the sum of these terms are zero.

So, it doesn't look that an integration is necessarily if the following equations are used:

\[
W = F_1 \cdot G_1 \quad (x>a \quad ; \quad y>b)
\]
\[
= F_2 \cdot G_1 \quad (0<x<a \quad ; \quad y>b)
\]
\[
= F_2 \cdot G_2 \quad (0<x<a \quad ; \quad 0<y<b)
\]
\[
= F_1 \cdot G_2 \quad (x>a \quad ; \quad 0<y<b) \tag{94}
\]

\( F_1 \) and \( F_2 \) are given by equations 91 and 92. \( G_1 \) and \( G_2 \) are obtained by replacing \( x \) by \( y \), \( a \) by \( b \), \( z_1 \) by \( r_1 \) and \( z_2 \) by \( r_2 \) in equations 91 and 92.

However, the following conditions must be fulfilled in order that the obtained function is a solution of the differential equation.

\[
\begin{align*}
  z_1^2 + r_1^2 &= g \pm \sqrt{g^2 - 1} ; \quad z_1^2 + r_2^2 = g \mp \sqrt{g^2 - 1} \\
  z_2^2 + r_1^2 &= g \pm \sqrt{g^2 - 1} ; \quad z_2^2 + r_2^2 = g \mp \sqrt{g^2 - 1}
\end{align*} \tag{95}
\]
And this leads either to a restriction which only can be met for g=1 (implying \( z_1=-z_2 \) and \( r_1=r_2 \)) or to \( z_1 = -z_2 \). This last restriction can not be used while Re\{z;r\} must be larger than 0 [Re\{z\}>0; Re\{r\}>0]. If \( z_1, z_2 \) and \( r_1 \) are chosen the last sub-equation in equation 95 will lead to a new \( r_3 \) value. In return the introduction of \( r_3 \) will lead to \( z \) and \( r \) values. So, the integral can be seen as a summation of all possible combinations. It is an infinite summation because each new \( r \) or \( z \) value requires new \( r \) and \( z \) values. Due to the negative expentionally character of the functions the integrand value decreases fast. Therefore instead of a (numerical) integration from 0 to infinity an integration from 0 to 1000 is often enough for the required accuracy. The disadvantage of the closed integral solution is that in order to solve the problem of a finite plate with a boundary the solutions of the homogenous differential equation have also to be written as a closed integral form solution.

It should be noted that the following general expression is a valid solution of the homogenous differential equation.

\[
F = e^{\frac{z^X}{L}} \cdot e^{\frac{r^Y}{L}}; (x<x_0; y<y_0); \ z^2 + r^2 = g \pm \sqrt{g^2 - 1} \tag{96}
\]

The same is true for equation 97

\[
F = \left[ \frac{X}{Z} - \frac{Y}{r} \right] \cdot e^{\frac{z^X}{L}} \cdot e^{\frac{r^Y}{L}}; (x<x_0; y<y_0); \ z^2 + r^2 = g \pm \sqrt{g^2 - 1} \tag{97}
\]

This last equation might be of use for the yet unknown solution for the problem of two boundaries at \( x=d \) and \( y=e \) (corner loading).
ANNEX 3 "The General Case"

-Two semi-infinite plates with shear force and moment transfer

If the joint between two adherent plates is not completely cracked transfer of moment is also possible at the 'joint'. From the point of view for the equilibrium of forces and moments we only have two boundary requirements. The first one for the transfer of moments \((M_1=M_2; \text{ equation 98})\) and the second one for the transfer of the shear force \(T\) in the plate and the shear force \(S\) in the foundation \((T_1+S_1=T_2+S_2; \text{ equation 99})\).

\[
- N. \left[ \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right] W_{x1d} = - N. \left[ \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right] W_{x1d} \tag{98}
\]

\[
- N. \left[ \frac{\partial^3}{\partial x^3} + (2 - \mu) \frac{\partial^3}{\partial x \partial y^2} - \frac{2g}{L^2} \frac{\partial}{\partial x} \right] W = \text{continuous at } x = d \tag{99}
\]

In order to solve the problem we do need two extra boundary equations. In principle any boundary restriction or equation is possible as long as it does not conflict the two boundary requirements for the equilibrium of forces and moments (equations 98 and 99). In this document we will only deal with boundary equations based on the deflections and the derivatives of the deflections at the joint. Three main types will be distinguished.

**Type I**

The two extra boundary equations are directly based on the deflections and the first derivatives (equations 100 and 101).

\[
\gamma \cdot W_{x1d} = W_{x1d} \tag{100}
\]

\[
\delta \cdot \frac{\partial W_{x1d}}{\partial x} = \frac{\partial W_{x1d}}{\partial x} \tag{101}
\]

Special cases are:

**I-1. \( \gamma \neq 1 \text{ and } \delta = 0 \)**

Because the first derivative at the right side of the joint is zero, also the shear force \(S_2\) will be zero. Therefore shear force transfer will only take place through a physical contact between the two plates.

**I-2. \( \gamma \neq 1 \text{ and } \delta = 1 \)**

In this case the shear forces \(S_1\) and \(S_2\) will be equal. This also implicates equality in the shear forces \(T_1\) and \(T_2\). The boundary conditions are therefore equal to the set: \(S_1=S_2; T_1=T_2; M_1=M_2\) and \(W_2=\gamma \cdot W_1\). The set of boundary conditions with \(\gamma=0\) is quite hypothetically.

**I-3. \( \gamma = 1 \text{ and } \delta = 1 \)**

This is the problem of an infinite plate. The (pseudo) boundary does not affect the behaviour of the system.
Type II

The two extra boundary conditions are related to the transfer of shear forces and moments according to equations 102 and 103.

\[
\frac{N}{L^3}(1-\alpha)[W_1 - W_2] = -N.\alpha \left( \frac{\partial^3}{\partial x^3} + (2-\mu) \frac{\partial^3}{\partial x \partial y \partial x} - \frac{2g}{L^2} \frac{\partial}{\partial x} \right) W_{x-d} \tag{102}
\]

\[
\frac{N}{L}(1-\beta) \left[ \frac{\partial}{\partial x} W_1 - \frac{\partial}{\partial x} W_2 \right]_{x=d} = -N.\beta \left[ \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right] W \tag{103}
\]

Special cases are:

**II-1.** \(\beta=1\)

This is the condition for which no transfer of moment occurs. If \(\alpha\) equals zero the deflections at both sides of the joint will be equal (Kerr’s requirement). If \(\alpha\) equals 1 the complete free boundary is obtained (no transfer of shear forces).

**II-2.** \(\beta=0\)

In this case the first derivatives are equal implying that also the shear forces \(S\) in the shear layer are equal. For \(\alpha=0\) the system of an infinite plate is obtained and for \(\alpha=1\) a very special case is obtained in which the moments are equal \((M_1=M_2)\) and the shear forces in the plate and shear layer are equal but opposite in sign \((T_1=-T_2=S_1=-S_2)\).

Type III

Type III is a variation of type I. It combines the advantages of the boundary equation for the deflections of type I and a boundary equation which describes the amount of moment transfer MTC.

\[
\gamma \cdot W_{x=1d} = W_{x=1d} \tag{104}
\]

\[
MTC = \beta = \frac{\left[ \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right] W_{x=d;\text{finite~plate}}}{\left[ \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} \right] W_{x=d;\text{infinite~plate}}} \tag{105}
\]

Taking \(\beta=0\) will give the free plate condition \((M=0)\). For \(\gamma=1\) the conditions are obtained in which the boundary does not have an effect on the integrity of the shear layer \((W_1=W_2;\text{ Kerr’s postulation})\). For \(\beta=0\) and \(\gamma=1\) the infinite plate is obtained and for \(\beta=1\) and \(\gamma=0\) the complete free system is obtained (no transfer trough plate or foundation).