Equations for the problem of a thin finite plate on a Pasternak foundation

Part 3: Homogenous Differential Equation
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on a Pasternak foundation

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Part 3: Homogenous Differential Equation

Introduction

In concrete pavement design the pavement is often regarded as or simulated by a thin plate resting on an elastic foundation. In the past the Winkler foundation was the commonly used model for the description of the foundation response. Nowadays the Pasternak foundation, a two parameter model, is a better simulation for the response. It allows for shear force transfer through the foundation which is not possible with the Winkler foundation (uncoupled vertical spring system). For this type of foundations in combination of a thin plate simulation for the concrete layers on top the big advantage is that stresses and strains can be calculate at the boundary of a finite plate.

However, the appearance of boundaries forces the search for suitable solutions of the homogenous differential equation in order to satisfy the requirements at the boundary. In part 1 and 2 we have dealt with the case of one boundary at the line $x=d$. In this document we will give an overview of valid solutions of the homogenous differential equation in the form of closed integrals. The homogenous differential equation is given by:

$$N \Delta \Delta \{W\} - G \Delta \{W\} + k \cdot W = 0$$  

(1)

$W$: Deflection at $x,y$  

$[m]$  

$k$: Modulus of subgrade reaction ($= N/L^4$)  

$[N/m^3]$  

$G$: Shear modulus for foundation ($= 2gl^2 = 2gN/L^2$)  

$[N/m]$  

$N$: Bending stiffness parameter ($= kL^4$)  

$[Nm]$  

Thin Plate on a Pasternak Foundation

Before going on we will repeat the solution of an infinite plate as given in W-DWW-98-010 which carries a rectangular load with its centre at $x=0$, $y=0$. For a rectangular load with sides $2a$ and $2b$ and a homogenous stress distribution $p$ the solution for an infinite plate is given by equation 2.

$$\frac{4pN}{\pi^2} \int_0^\infty \int_0^\infty \frac{\cos \left( \frac{x^t}{L} \right) \sin \left( \frac{a^t}{L} \right) \cos \left( \frac{y^s}{L} \right) \sin \left( \frac{b^s}{L} \right)}{ts \left( t^4 + 2t^2s^2 + 2gt^2 + 2gs^2 + s^4 + 1 \right)} \, dt \, ds$$

If the denominator is split into three terms (with respect to $t$) as given in Annex 1 of W-DWW-98-010, this double integral can be simplified into a single integral of the form:

$$\frac{2p}{\pi k} \int_0^\infty \frac{\cos \left( \frac{y^s}{L} \right) \sin \left( \frac{b^s}{L} \right)}{s} \times \left( F_1 \{s\} + F_2 \{x,s\} + F_3 \{x,s\} \right) ds$$  

(3)
The function $F_1(s)$ is zero for $x>a$ and equals one for $x<a$. Using the same procedure the integral which contains the function $F_1(s)$ can be split into three terms of which the first is of the form as given in equation 4:

$$\frac{2p}{\pi k} \int_0^\infty \left[ \frac{\cos(y \frac{s}{L}) \sin(b \frac{s}{L})}{s} \right] x(F_1^* = 1 (x < a) \text{ or } 0 (x > a)) ds$$

$$= \frac{p}{k} \text{ if } y < b \quad \text{or} \quad 0 \text{ if } y > b$$

Therefore the solution can be considered as the combination of the particular solution ($W=p/k$ inside the loaded area and $W=0$ outside the loaded area) and suitable solutions of the homogenous differential equation which satisfy the requirements at the pseudo boundaries of the loaded area ($|x|=a; \ |y|=b$). The word suitable is used because these functions changes of formulation at the pseudo boundaries. Mark that the particular solution is obtained by or due to the term $t^s$’s in the original formulation of the solution. For convenience we will repeat Annex 2 from reference W-DWW-98-010.

""" The pseudo boundary is a boundary at which the load on the plate or the formulation for the deflection changes but no changes occur in the response formulation for the behaviour of plate and foundation. Therefore there exists no real boundary while the deflection W, the first derivative W' (normal to the pseudo boundary), the moment M and the shear force T+S must be continuous at this crossing. Observing the equations involved this means that the function W itself and its first, second and third derivatives normal to the pseudo boundary must be continuous. We will show this for the solution of the loaded infinite plate at the crossing $x=a$ ($x>b$). The integrand value (containing the dependency on $x$) at the right side of $x=a$ is given by equation 5 and at the left side of $x=a$ by equation 6:

$$F_{w;x>a} = -B_s \cdot e^{-z_1 \frac{x}{L}} \cdot \sinh(z_1 \frac{a}{L}) - C_s \cdot e^{-z_2 \frac{x}{L}} \cdot \sinh(z_2 \frac{a}{L})$$  (5)

$$F_{w;0<x<a} = A_s + B_s \cdot e^{-z_1 \frac{x}{L}} \cdot \cosh(z_1 \frac{x}{L}) + C_s \cdot e^{-z_2 \frac{x}{L}} \cdot \cosh(z_2 \frac{x}{L})$$  (6)

$$A_s = \frac{1}{z_1^2 z_2^2} \quad ; \quad B_s = \frac{1}{z_1^2 z_1^2 - z_2^2} \quad ; \quad C_s = -\frac{1}{z_2^2 z_1^2 - z_2^2}$$  (7)

The function value is continuously [$A_s + (B_s + C_s)/2 = -(B_s + C_s)/2$]. For the first and third derivatives the separate terms (containing the coefficients $B_s$ and $C_s$) are continuously while for the second derivative the sum of these terms are zero.
So, it doesn't look that an integration is necessarily if the following equations are used:

\[
W = F_1 \cdot G_1 \quad (x>a \quad ; y>b) \\
= F_2 \cdot G_1 \quad (0<x<a \quad ; y>b) \\
= F_2 \cdot G_2 \quad (0<x<a \quad ; 0<y<b) \\
= F_1 \cdot G_2 \quad (x>a \quad ; 0<y<b)
\]

(8)

F_1 and F_2 are given by equations 5 and 6. G_1 and G_2 are obtained by replacing x by y, a by b, z_1 by v_1 and z_2 by v_2 in equations 5 and 6.

However, the following conditions must be fulfilled in order that the obtained function is a solution of the differential equation.

\[
z_1^2 + v_1^2 = g \pm \sqrt{g^2 - 1} \quad ; \quad z_1^2 + v_2^2 = g \pm \sqrt{g^2 - 1}
\]

\[
z_2^2 + v_1^2 = g \pm \sqrt{g^2 - 1} \quad ; \quad z_2^2 + v_2^2 = g \pm \sqrt{g^2 - 1}
\]

(9)

And this leads either to a restriction which only can be met for g=1 (implying z_1=z_2 and v_1=v_2) or to z_1 = - z_2. This last restriction can not be used while Re[z; r] must be larger than 0 [Re[z]>0; Re[v]>0]. If z_1, z_2 and v_1 are chosen the last sub-equation in equation 9 will lead to a new r_3 value. In return the introduction of r_3 will lead to z and v values. So, the integral can be seen as a summation of all possible combinations. It is an infinite summation because each new r or z value requires new v and z values. Due to the negative exponentially character of the functions the integrand value decreases fast. Therefore instead of a (numerical) integration from 0 to infinity an integration from 0 to 1000 is often enough for the required accuracy. The disadvantage of the closed integral solution is that in order to solve the problem of a finite plate with a boundary the solutions of the homogenous differential equation have also to be written as a closed integral form solution.

It should be noted that the following general expression is a valid solution of the homogenous differential equation.

\[
F = e^{z^x \over L} \cdot e^{v^x \over L} \quad ; \quad (x<x_0 \quad ; y<y_0) \quad ; \quad z^2 + v^2 = g \pm \sqrt{g^2 - 1}
\]

(10)

The same is true for equation 11

\[
F = \left[ {x \over z} - {y \over v} \right] \cdot e^{z^x \over L} \cdot e^{v^x \over L} \quad ; \quad (x<x_0 \quad ; y<y_0) \quad ; \quad z^2 + v^2 = g \pm \sqrt{g^2 - 1}
\]

(11)

This last equation might be of use for the yet unknown solution for the problem of two boundaries at x=d and y=e (corner loading).
The closed integral form creates the opportunity to obtain in a direct way all the possible solutions by a "coupling" between the integration 'variables' s and t in the denominator part as for example given in equation 12 for the solution of the loaded infinite plate:

$$ t^4 + 2t^2s^2 + 2gt^2 + 2gs^2 + s^4 + 1 $$

This denominator is split into two parts for the integration with respect to the dummy t. The denominators of the obtained integrals are of the form as given in equation 13:

$$ t^2 + z_1^2 ; \quad t^2 + z_2^2 \quad z_{1,2} = s^2 + g \pm \sqrt{g^2 - 1} $$

(13)

(It is also valid to start with the integration with respect to the dummy s).

The integration with respect to t should lead to an exponential function in x of the form:

$$ A[z_i] \cdot e^{\frac{z_1 x}{\lambda}} $$

(14)

The remaining integration with respect to s should contain a (simple) trigonometric function of y. In this way the complete integral will satisfies the differential equation, because the following condition is fulfilled.

$$ \Delta F(x, y) = (z_{1,2}^2 - s^2).F = (g \pm \sqrt{g^2 - 1}).F $$

(15)

In Annex 1 a limited overview is given of integrals which all will lead to the required exponentially functions.

However, it should be noted that the pseudo boundaries at the loaded area (|x|<a;|y|<b) do have an influence on the suitable/required solutions of the homogenous differential equation. If one real boundary is present at x=d>a the required solutions of the homogenous differential equation are obtained by an integration with respect to t. In this way a single closed form integral is obtained of the homogenous differential equation (F) which changes of form at the pseudo boundaries y = + b and y = - b. If a second boundary is present at y = e>b the double integral for the solution of the infinite loaded plate should be integrated with respect to s leading to a single closed form integral which changes of form at the pseudo boundaries x = + a and x = - a. In the same way as before (see W-DWW-98-010 and W-DWW-98-017) the required solutions (G) for the homogenous differential equations can be obtained which satisfies the boundary requirements at y = e. The problem is now "reduced" to find solutions (H and I) which satisfy the boundary requirements for the functions F at y = e and G at x = d. Taking into account the different formulations of the single closed form integrals the following suitable match of equations can be obtained.

<table>
<thead>
<tr>
<th>Boundary requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>x=d \quad W_\infty + F (known)</td>
</tr>
<tr>
<td>y=e \quad W_\infty + G (known)</td>
</tr>
</tbody>
</table>

The functions $W_\infty$, F and G are known. The functions $H_{1,2}$ and $I_{1,2}$ have to be established.
Solutions of the homogeneous differential equation

The homogenous differential equation is rewritten as:

$$\left[ \frac{N}{k} = L^4 \right] \Delta \Delta \{W\} - \left[ \frac{G}{k} = 2gL^2 \right] \Delta \{W\} + W = 0 \tag{16}$$

The following abbreviations are used:

$$E_1 = e^{z_1 \left( \frac{x-d}{L} \right)}; \quad E_2 = e^{z_2 \left( \frac{x-d}{L} \right)}; \quad \text{Re}\{z_1; z_2\} > 0$$

$$Q_1 = \frac{z_1^2 - s^2}{L^2} = g + \sqrt{g^2 - 1}; \quad Q_2 = \frac{z_2^2 - s^2}{L^2} = g - \sqrt{g^2 - 1} \tag{17}$$

Each function F which fulfills the requirement of equation 18 is a valid solution of equation 16.

$$\Delta \{F\} = Q_1 . F; \quad \Delta \{F\} = Q_2 . F \tag{18}$$

$$L^4 \Delta \{\Delta \{F\}\} - 2gL^2 \Delta \{F\} + F = \left[ L^4 . Q_{1,2} - 2gL^2 . Q_{1,2} + 1 \right] . F = \left[ g^2 + 2g\sqrt{g^2 - 1} + g^2 - 1 - 2g \left( g \pm \sqrt{g^2 - 1} \right) + 1 \right] . F = 0 . F = 0 \tag{19}$$

First we will deal with the following function:

$$F = A_{x_1} \cdot \frac{X}{L} \cdot \cos(s \frac{Y}{L}) \cdot e^{z_1 \left( \frac{x-d}{L} \right)} = A_{x_1} \cdot \frac{X}{L} \cdot \cos(s \frac{Y}{L}) . E_1 \tag{20}$$

$$F_x = \left[ \frac{1}{L} . A_{x_1} \cdot \cos(s \frac{Y}{L}) + \frac{Z_1}{L} . A_{x_1} \cdot \frac{X}{L} \cdot \cos(s \frac{Y}{L}) \right] E_1 \tag{21}$$

$$F_{xx} = \left[ \frac{2Z_1}{L^2} . A_{x_1} \cdot \cos(s \frac{Y}{L}) + \frac{Z_1^2}{L^2} . A_{x_1} \cdot \frac{X}{L} \cdot \cos(s \frac{Y}{L}) \right] E_1 \tag{22}$$

$$F_{yy} = \left[ -\frac{s^2}{L^2} . A_{x_1} \cdot \frac{X}{L} \cdot \cos(s \frac{Y}{L}) \right] E_1 \tag{23}$$

$$\Delta \{F\} = Q_1 . F + \frac{2Z_1}{L^2} . A_{x_1} \cdot \cos(s \frac{Y}{L}) . E_1 \tag{24}$$
\[ \Delta(\Delta \{F\}) = Q_1^2 \cdot F + \frac{4z_1}{L^2} \cdot Q_1 \cdot A_{x_1} \cdot \cos(s \frac{Y}{L}) \cdot E_1 \] (25)

The function F in itself is not a solution of the homogenous differential equation. There is a remaining part (equation 26) which is not equal to zero.

\[ 4z_1 \cdot A_{x_1} \cdot \left[ L^2 \cdot Q_1 - g \right] \cdot \cos(s \frac{Y}{L}) \cdot E_1 \neq 0 \quad \text{for } g \neq 1 \] (26)

But we can adopt also the function G of equation 27.

\[ G = A_{y_1} \cdot \frac{Y}{L} \cdot \sin(s \frac{Y}{L}) \cdot e^{\frac{z_1}{L}(x-d)} = A_{y_1} \cdot \frac{Y}{L} \cdot \sin(s \frac{Y}{L}) \cdot E_1 \] (27)

\[ G_y = \left[ \frac{1}{L} \cdot A_{y_1} \cdot \sin(s \frac{Y}{L}) + \frac{S}{L^2} \cdot A_{y_1} \cdot \frac{Y}{L} \cdot \cos(s \frac{Y}{L}) \right] E_1 \] (28)

\[ G_{yy} = \left[ -\frac{2s}{L^2} \cdot A_{y_1} \cdot \cos(s \frac{Y}{L}) - \frac{s^2}{L^2} \cdot A_{y_1} \cdot \frac{Y}{L} \cdot \sin(s \frac{Y}{L}) \right] E_1 \] (29)

\[ G_{xx} = \left[ + \frac{Z_1^2}{L^2} \cdot A_{y_1} \cdot \frac{Y}{L} \cdot \sin(s \frac{Y}{L}) \right] E_1 \] (30)

\[ \Delta \{G\} = Q_1 \cdot G + \frac{2s}{L^2} \cdot A_{y_1} \cdot \cos(s \frac{Y}{L}) \cdot E_1 \] (31)

\[ \Delta(\Delta \{G\}) = Q_1^2 \cdot G + \frac{4s}{L^2} \cdot Q_1 \cdot A_{y_1} \cdot \cos(s \frac{Y}{L}) \cdot E_1 \] (32)

The remaining part is now equal to:

\[ 4s \cdot A_{y_1} \cdot \left[ L^2 \cdot Q_1 - g \right] \cdot \cos(s \frac{Y}{L}) \cdot E_1 \neq 0 \quad \text{for } g \neq 1 \] (33)

Therefore if choose \( A_{x_1} = +A_y/z_1 \) and \( A_{y_1} = -A_y/s \) the function F+G will be an acceptable solution of the homogenous differential equation.

\[ F_1 = A_{1} \left[ \frac{1}{Z_1} \cdot \frac{X}{L} \cdot \cos(s \frac{Y}{L}) - \frac{1}{s} \cdot \frac{Y}{L} \cdot \sin(s \frac{Y}{L}) \right] E_1 \] (34)
The same procedure can be followed for the following functions:

\[ F = B_{x_1} \cdot \frac{x}{L} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \cdot e^{x \cdot z_1 \left(\frac{x - d}{L}\right)} = B_{x_1} \cdot \frac{x}{L} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \cdot E_1 \]  

\[ F_x = \left[ \frac{1}{L} \cdot B_{x_1} \cdot \sin\left(\frac{s \cdot Y}{L}\right) + \frac{z_1}{L} \cdot B_{x_1} \cdot \frac{x}{L} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \right] E_1 \]  

\[ F_{xx} = \left[ \frac{2}{L^2} \cdot B_{x_1} \cdot \sin\left(\frac{s \cdot Y}{L}\right) + \frac{z_1^2}{L^2} \cdot B_{x_1} \cdot \frac{x}{L} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \right] E_1 \]  

\[ F_{yy} = \left[ -\frac{s^2}{L^2} \cdot B_{x_1} \cdot \frac{x}{L} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \right] E_1 \]  

\[ \Delta\{F\} = Q_1 \cdot F + \frac{2}{L^2} \cdot B_{x_1} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \cdot E_1 \]  

\[ \Delta\{\Delta\{F\}\} = Q_1^2 \cdot F + \frac{4}{L^2} \cdot Q_1 \cdot B_{x_1} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \cdot E_1 \]  

\[ 4 \cdot z_1 \cdot B_{x_1} \cdot \left[ L^2 \cdot Q_1 - g \right] \cdot \sin\left(\frac{s \cdot Y}{L}\right) \cdot E_1 \neq 0 \; \text{for} \; g \neq 1 \]  

\[ G = B_{y_1} \cdot \frac{Y}{L} \cdot \cos\left(\frac{s \cdot Y}{L}\right) \cdot e^{x \cdot z_1 \left(\frac{x - d}{L}\right)} = B_{y_1} \cdot \frac{Y}{L} \cdot \cos\left(\frac{s \cdot Y}{L}\right) \cdot E_1 \]  

\[ G_y = \left[ \frac{1}{L} \cdot B_{y_1} \cdot \cos\left(\frac{s \cdot Y}{L}\right) - \frac{s}{L} \cdot B_{y_1} \cdot \frac{Y}{L} \cdot \sin\left(\frac{s \cdot Y}{L}\right) \right] E_1 \]  

\[ G_{yy} = \left[ -\frac{2}{L^2} \cdot B_{y_1} \cdot \sin\left(\frac{s \cdot Y}{L}\right) - \frac{s^2}{L^2} \cdot B_{y_1} \cdot \frac{Y}{L} \cdot \cos\left(\frac{s \cdot Y}{L}\right) \right] E_1 \]  

\[ G_{xx} = \left[ +\frac{z_1^2}{L^2} \cdot B_{y_1} \cdot \frac{Y}{L} \cdot \cos\left(\frac{s \cdot Y}{L}\right) \right] E_1 \]
\[
\Delta \{G\} = Q_1 \cdot G - \frac{2s}{L^2} B_{y1} \cdot \sin \left(\frac{sY}{L}\right) \cdot E_1
\]  
(46)

\[
\Delta \{\Delta \{G\}\} = Q_1^2 \cdot G - \frac{4s}{L^2} Q_1 \cdot B_{y1} \cdot \sin \left(\frac{sY}{L}\right) \cdot E_1
\]  
(47)

The remaining part is now equal to:

\[
-4s \cdot B_{y1} \cdot \left[ L^2 \cdot Q_1 - g \right] \cdot \sin \left(\frac{sY}{L}\right) \cdot E_1 \neq 0 \quad \text{for } g \neq 1
\]  
(48)

Therefore if choose \(B_{y1} = +B_z/z_1\) and \(B_{y1} = +B_z/s\) the function \(F+G\) will be an acceptable solution of the homogenous differential equation.

\[
G_1 = B_{y1} \cdot \left[ \frac{1}{z_1} \cdot \frac{X}{L} \cdot \sin \left(\frac{sY}{L}\right) + \frac{1}{s} \cdot \frac{Y}{L} \cdot \cos \left(\frac{sY}{L}\right) \right] \cdot E_1
\]  
(49)

The also valid equations \(F_2\) and \(G_2\) are obtained by replacing \(z_1\) by \(z_2\), \(A_1\) by \(A_2\), \(B_1\) by \(B_2\) and \(E_1\) by \(E_2\).

It should be marked that the following (principal) functions are of course also valid solutions of the homogenous differential equation.

\[
W_1 = C_1 \cdot \cos \left(\frac{sY}{L}\right) \cdot E_1 \quad ; \quad W_2 = C_2 \cdot \cos \left(\frac{sY}{L}\right) \cdot E_2
\]  
(50)

\[
V_1 = D_1 \cdot \sin \left(\frac{sY}{L}\right) \cdot E_1 \quad ; \quad V_2 = D_2 \cdot \sin \left(\frac{sY}{L}\right) \cdot E_2
\]  
(51)

### Boundary Conditions

For a limited plate on a foundation one needs the boundary requirements in order to solve the problem. In this paper we will not deal with the formulation of the boundary requirements but just checking if it is possible to solve in a symbolic analytical way the problem of two boundaries \((x=d \text{ and } y=e)\). In fact we will analyze if the problem can be solved by finding enough and applicable solutions of the homogeneous differential equation. For that reason we will discuss the requirements for a limited slab with no foundation outside the slab (no moment and shear force transfer at the boundaries).

In the case of no moment transfer at the boundary \(x=d\) \((E_1=E_2=1)\) and applying the functions \(F_1\) and \(F_2\) the "unwanted" terms in the moment equation: the terms which contains \(y \cdot \sin(sy/L)\) will vanish if \(A_1 = + (z_2^2-\mu s^2)A = u_2A\) and \(A_2 = - (z_1^2-\mu s^2)A = - u_1A\). For the functions \(G_1\) and \(G_2\) there are two "unwanted" terms: \(\sin(sy/L)\) and \(y \cdot \cos(sy/L)\). The second type of terms will vanish for \(B_1 = u_2B\) and \(B_2 = u_1B\) but not the terms of the first type: \(\sin(sy/L)\). The \(\sin(sy/L)\) terms can be removed by using the functions \(V_1\) and \(V_2\).
Review for the moment condition $M=0$ at $x=d$ ($E_1=E_2=1$)

At the boundary $x=d$ the equations for the moment normal to boundary will consist out of the following terms:
1) $(x=d)\cdot \cos(sy/L)$ and $\cos(sy/L)$
2) $(x=d)\cdot \sin(sy/L)$ and $\sin(sy/L)$
3) $y^*\sin(sy/L)$
4) $y^*\cos(sy/L)$

Thus it seems that four equations are needed.
The first equation will contain the coefficients: $C_1$, $C_2$, $A_1$, $A_2$ and of course $W_\infty$
The second one will contain the coefficients: $D_1$, $D_2$, $B_1$ and $B_2$
The third one will contain the coefficients: $A_1$ and $A_2$
The fourth one will contain the coefficients: $B_1$ and $B_2$

Boundary condition $M=0$ at $y=e$ ($E_1 \neq E_2$)

Now we will involve the boundary condition at $y=e$ in the case of no moment transfer ($M=0$). Because the $x$ coordinate varies along this boundary the following distinction have to be made:
5) $\sin(s[y=e]/L)\cdot E_1$ and $\cos(s[y=e]/L)\cdot E_1$ $\rightarrow C_1$, $D_1$, $A_1$ and $B_1$
6) $\sin(s[y=e]/L)\cdot E_2$ and $\cos(s[y=e]/L)\cdot E_2$ $\rightarrow C_2$, $D_2$, $A_2$ and $B_2$
7) $x^\cdot \sin(s[y=e]/L)\cdot E_1$ and $x^\cdot \cos(s[y=e]/L)\cdot E_1$ $\rightarrow A_1$ and $B_1$
8) $x^\cdot \sin(s[y=e]/L)\cdot E_2$ and $x^\cdot \cos(s[y=e]/L)\cdot E_2$ $\rightarrow A_2$ and $B_2$

From this point of view it looks like there is no 'freedom' left to satisfy the boundary conditions for the shear force at the two boundaries (8 equations and 8 coefficients).
We will deal with this aspect in part 4 "The Final Quest: Corner Loading" of this series of notes and show that it will be possible to find the solution for corner loading in this way.

It is possible to create another four solutions of the homogenous differential equation by a suitable combination of integrals which contains the terms $x^2$, $xy$ and $y^2$. However the number of boundary equations will also increase by four $(y^*\sin(sy/L)$, $y^*\cos(sy/L)$, $x^2E_1$ and $x^2E_2$). Therefore increasing the power of $x$ and $y$ will not lead to a solution of the corner load problem. However, the integrals 56 to 59 (Annex 1) might be very useful. By a suitable combinations these integrals will become solutions of the homogenous differential equation and still no extra boundary conditions containing powers of $x$ or $y$ will appear.

By integration of the original (infinite) double integral solution with respect to $s$ we will obtain the solution with cosines terms for $x$ and exponentially terms for $y$. The type of form is however identical. Thus the same procedure can be repeated. This will lead to 4 boundary equations at $y=e$ in which the exponentially terms are equal to 1 and four boundary equations in which the cosines and sinus terms can be combined.
ANNEX 1

Remark: In the following integrals the variable $x$ can be replaced by $d-x$ with $x<d$ or $x-d$ for $x>d$. In this way integrals are obtained for the interval $-\infty$ to $d$ (and for $d$ to $+\infty$ which still fulfil the requirements with respect to the convergence of the integral.

Overview of definite integrals

\[
\int_0^\infty \frac{\cos(t\frac{x}{L})}{t^2 + z_1^2} \, dt = \frac{\pi}{2 z_1} \cdot e^{-z_1\frac{x}{L}}
\] (52)

\[
\int_0^\infty \frac{t \sin(t\frac{x}{L})}{t^2 + z_1^2} \, dt = \frac{\pi}{2} \cdot e^{-z_1\frac{x}{L}}
\] (53)

\[
\int_0^\infty \frac{\sin(t\frac{x}{L})}{t \cdot (t^2 + z_1^2)} \, dt = \frac{\pi}{2 z_1^2} \left[ 1 - e^{-z_1\frac{x}{L}} \right]
\] (54)

\[
\int_0^\infty \frac{\sin(t\frac{x}{L})}{t \cdot z_1^2} \, dt = \frac{\pi}{2 z_1^2} \cdot \text{SIGN} \left( \frac{x}{L} \right)
\] (55)

\[
\int_0^\infty \frac{\cos(t\frac{x}{L})}{z_1^3} \cdot e^{-z_1\frac{x}{L}} \cdot \left[ \cos \left( \frac{z_1}{\sqrt{2}} \cdot \frac{x}{L} \right) + \sin \left( \frac{z_1}{\sqrt{2}} \cdot \frac{x}{L} \right) \right] \, dt
\] (56)

\[
\int_0^\infty \frac{t \sin(t\frac{x}{L})}{t^4 + z_1^4} \, dt = \frac{\pi}{2 z_1^2} \cdot e^{-z_1\frac{x}{L}} \cdot \sin \left( \frac{z_1}{\sqrt{2}} \cdot \frac{x}{L} \right)
\] (57)

\[ \int_0^\infty \frac{t^2 \cos\left(\frac{X}{L}\right)}{t^4 + z_1^4} \, dt = \frac{\pi}{2} e^{-\frac{z_1}{\sqrt{2} \cdot L}} \cdot \cos\left(\frac{z_1}{\sqrt{2} \cdot L} \cdot \frac{X}{L}\right) - \sin\left(\frac{z_1}{\sqrt{2} \cdot L} \cdot \frac{X}{L}\right) \]  
(58)

\[ \int_0^\infty \frac{t^3 \sin\left(\frac{X}{L}\right)}{t^4 + z_1^4} \, dt = \frac{\pi}{2} e^{-\frac{z_1}{\sqrt{2} \cdot L}} \cdot \cos\left(\frac{z_1}{\sqrt{2} \cdot L} \cdot \frac{X}{L}\right) \]  
(59)

\[ \int_0^\infty \frac{\cos\left(\frac{X}{L}\right)}{(t^2 + z_1^2) \cdot (t^2 + z_2^2)} \, dt = \frac{\pi}{2 z_1 z_2 (z_1^2 - z_2^2)} \cdot \left[ z_1 e^{-\frac{z_2}{L} \cdot \frac{X}{L}} - z_2 e^{-\frac{z_1}{L} \cdot \frac{X}{L}} \right] \]  
(60)

\[ \int_0^\infty \frac{t \sin\left(\frac{X}{L}\right)}{(t^2 + z_1^2) \cdot (t^2 + z_2^2)} \, dt = \frac{\pi}{2 (z_2^2 - z_1^2)} \left[ e^{-\frac{z_1}{L} \cdot \frac{X}{L}} - e^{-\frac{z_2}{L} \cdot \frac{X}{L}} \right] \]  
(61)

\[ \int_0^\infty \frac{t^2 \cos\left(\frac{X}{L}\right)}{(t^2 + z_1^2) \cdot (t^2 + z_2^2)} \, dt = \frac{\pi}{2 (z_1^2 - z_2^2)} \cdot \left[ z_1 e^{-\frac{z_2}{L} \cdot \frac{X}{L}} - z_2 e^{-\frac{z_1}{L} \cdot \frac{X}{L}} \right] \]  
(62)

\[ \int_0^\infty \frac{t^3 \sin\left(\frac{X}{L}\right)}{(t^2 + z_1^2) \cdot (t^2 + z_2^2)} \, dt = \frac{\pi}{2 (z_2^2 - z_1^2)} \left[ z_1^2 e^{-\frac{z_1}{L} \cdot \frac{X}{L}} - z_2^2 e^{-\frac{z_2}{L} \cdot \frac{X}{L}} \right] \]  
(63)
\[ \int_{0}^{\infty} e^{-\beta \sqrt{s^2 + \gamma^2}} \cdot \cos(s \cdot x) \cdot ds = \frac{\beta \cdot \gamma}{\sqrt{\beta^2 + x^2}} \cdot K_1\left(\gamma \cdot \sqrt{\beta^2 + x^2}\right) \quad (64) \]

\[ \int_{0}^{\infty} \frac{s \cdot e^{-\beta \sqrt{s^2 + \gamma^2}}}{s^2 + \gamma^2} \cdot \sin(s \cdot x) \cdot ds = \frac{x \cdot \gamma}{\sqrt{\beta^2 + x^2}} \cdot K_1\left(\gamma \cdot \sqrt{\beta^2 + x^2}\right) \quad (65) \]

\[ \int_{0}^{\infty} \frac{e^{-\beta \sqrt{s^2 + \gamma^2}}}{\sqrt{s^2 + \gamma^2}} \cdot \cos(s \cdot x) \cdot ds = K_0\left(\gamma \cdot \sqrt{\beta^2 + x^2}\right) \quad (66) \]

\[ \int_{0}^{\infty} \frac{\sqrt{s^2 + \gamma^2 - \gamma \cdot e^{-\beta \sqrt{s^2 + \gamma^2}}}}{\sqrt{s^2 + \gamma^2}} \cdot \sin(s \cdot x) \cdot ds = \]

\[ \sqrt{\frac{\pi}{2}} \cdot \frac{x \cdot e^{-\gamma \cdot \sqrt{\beta^2 + x^2}}}{\sqrt{\beta^2 + x^2} \cdot \sqrt{\beta^2 + x^2 + \beta}} \quad (67) \]

\[ \int_{0}^{\infty} \frac{s \cdot e^{-\beta \sqrt{s^2 + \gamma^2}}}{\sqrt{s^2 + \gamma^2} \cdot \sqrt{s^2 + \gamma^2 - \gamma}} \cdot \cos(s \cdot x) \cdot ds = \]

\[ \sqrt{\frac{\pi}{2}} \cdot \frac{\sqrt{\beta^2 + x^2 + \beta} \cdot e^{-\gamma \cdot \sqrt{\beta^2 + x^2}}}{\sqrt{\beta^2 + x^2}} \quad (68) \]

\[ \int_{0}^{\infty} \frac{e^{-\beta \sqrt{s^4 + \gamma^4}}}{\sqrt{s^4 + \gamma^4}} \cdot \sin(x \cdot s^2) \cdot ds = \]

\[ \sqrt{\frac{x \cdot \pi}{8}} \cdot \frac{1}{4} \left\{ \frac{\gamma^2}{2} \cdot \left(\sqrt{\beta^2 + x^2 - \beta}\right) \right\} \cdot K_{1/4}\left(\frac{\gamma^2}{4} \cdot \left(\sqrt{\beta^2 + x^2 + \beta}\right)\right) \quad (69) \]

For the two last formulas use the transformation \( t=s^2 \) and \( dt/ds=1/\{2\sqrt{t}\} \).
\[
\int_0^\infty \frac{e^{-\beta \sqrt{s^4 + \gamma^4}} \cos(x s^2)}{\sqrt{s^4 + \gamma^4}} \cdot ds = \sqrt{\frac{x \pi}{8}} J_{1/4} \left( \frac{\gamma^2}{2} \left( \sqrt{\beta^2 + x^2} - \beta \right) \right) K_{1/4} \left( \frac{\gamma^2}{4} \left( \sqrt{\beta^2 + x^2} + \beta \right) \right)
\]

ANNEX 2

Possible Functions H and I

\[
H_{0,0,c,z_1} = \int_0^\infty H_0(s) \cdot \cos \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]

\[
H_{1,0,c,z_1} = \int_0^\infty H_1(s) \cdot \frac{X}{L} \cdot \cos \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]

\[
H_{0,1,c,z_1} = \int_0^\infty H_2(s) \cdot \frac{Y}{L} \cdot \cos \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]

\[
H_{1,1,c,z_1} = \int_0^\infty H_3(s) \cdot \frac{X Y}{L^2} \cdot \cos \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]

\[
H_{0,0,s,z_1} = \int_0^\infty H_4(s) \cdot \sin \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]

\[
H_{1,0,s,z_1} = \int_0^\infty H_5(s) \cdot \frac{X}{L} \cdot \sin \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]

\[
H_{0,1,s,z_1} = \int_0^\infty H_6(s) \cdot \frac{Y}{L} \cdot \sin \left( s \frac{Y}{L} \right) \cdot e^{-z_1 \left( \frac{d-x}{L} \right)} \cdot ds
\]
\[ H_{1,1,s,z_1} = \int_0^\infty H_7(s). \frac{x \cdot y}{L^2} \cdot \sin(s \cdot \frac{y}{L}) \cdot e^{-z_1\left(\frac{d-x}{L}\right)} \cdot ds \]  
(78)

\[ H_{0,0,c,z_2} = \int_0^\infty H_8(s) \cdot \cos(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(79)

\[ H_{1,0,c,z_2} = \int_0^\infty H_9(s) \cdot \frac{x}{L} \cdot \cos(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(80)

\[ H_{0,1,c,z_2} = \int_0^\infty H_{10}(s) \cdot \frac{y}{L} \cdot \cos(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(81)

\[ H_{1,1,c,z_2} = \int_0^\infty H_{11}(s) \cdot \frac{x \cdot y}{L^2} \cdot \cos(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(82)

\[ H_{0,0,s,z_2} = \int_0^\infty H_{12}(s) \cdot \sin(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(83)

\[ H_{1,0,s,z_2} = \int_0^\infty H_{13}(s) \cdot \frac{x}{L} \cdot \sin(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(84)

\[ H_{0,1,s,z_2} = \int_0^\infty H_{14}(s) \cdot \frac{y}{L} \cdot \sin(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(85)

\[ H_{1,1,s,z_2} = \int_0^\infty H_{15}(s) \cdot \frac{x \cdot y}{L^2} \cdot \sin(s \cdot \frac{y}{L}) \cdot e^{-z_2\left(\frac{d-x}{L}\right)} \cdot ds \]  
(86)
\begin{align*}
I_{0,0,c,z_1} &= \int_0^\infty I_0(s) \cdot \cos\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{1,0,c,z_1} &= \int_0^\infty I_1(s) \cdot \frac{x}{L} \cdot \cos\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{0,1,c,z_1} &= \int_0^\infty I_2(s) \cdot \frac{y}{L} \cdot \cos\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{1,1,c,z_1} &= \int_0^\infty I_3(s) \cdot \frac{x \cdot y}{L^2} \cdot \cos\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{0,0,s,z_1} &= \int_0^\infty I_4(s) \cdot \sin\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{1,0,s,z_1} &= \int_0^\infty I_5(s) \cdot \frac{x}{L} \cdot \sin\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{0,1,s,z_1} &= \int_0^\infty I_6(s) \cdot \frac{y}{L} \cdot \sin\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds \\
I_{1,1,s,z_1} &= \int_0^\infty I_7(s) \cdot \frac{x \cdot y}{L^2} \cdot \sin\left(s \frac{x}{L}\right) \cdot e^{-z_1\left(\frac{e-y}{L}\right)} \cdot ds
\end{align*}
\[ I_{0,0,c,z_2} = \int_0^\infty I_8(s) \cdot \cos(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(95)

\[ I_{1,0,c,z_2} = \int_0^\infty I_9(s) \cdot \frac{x}{L} \cdot \cos(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(96)

\[ I_{0,1,c,z_2} = \int_0^\infty I_{10}(s) \cdot \frac{y}{L} \cdot \cos(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(97)

\[ I_{1,1,c,z_2} = \int_0^\infty I_{11}(s) \cdot \frac{x \cdot y}{L^2} \cdot \cos(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(98)

\[ I_{0,0,s,z_2} = \int_0^\infty I_{12}(s) \cdot \sin(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(99)

\[ I_{1,0,s,z_2} = \int_0^\infty I_{13}(s) \cdot \frac{x}{L} \cdot \sin(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(100)

\[ I_{0,1,s,z_2} = \int_0^\infty I_{14}(s) \cdot \frac{y}{L} \cdot \sin(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(101)

\[ I_{1,1,s,z_2} = \int_0^\infty I_{15}(s) \cdot \frac{x \cdot y}{L^2} \cdot \sin(s \frac{x}{L}) \cdot e^{-z_2 \left( \frac{a-y}{L} \right)} \cdot ds \]  
(102)