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ABSTRACT

A theoretically sound and practical approach is described for determining the maximum responses in concrete pavement systems incorporating a base layer. Equations are presented that may be used with either an elastic solid or a dense liquid foundation under any of the three fundamental loading conditions. These formulae are extensions of available closed-form solutions corrected for the compressions in the two placed layers which are ignored by plate theory. The proposed methodology may be easily incorporated in a personal computer spreadsheet or in a programmable calculator. Research activities for its full verification and refinement are continuing at this time. It is anticipated that such theoretically based investigations will encourage the elimination of theoretically questionable empirical concepts such as that of deriving a composite 'top-of-the-base' subgrade modulus.
STRUCTURAL EVALUATION OF BASE LAYERS IN CONCRETE PAVEMENT SYSTEMS

By Anastasios M. Ioannides¹, Lev Khazanovich² and Jennifer L. Becque³

INTRODUCTION

Conventional analysis and mechanistic-based design procedures for Portland Cement Concrete (PCC) pavement systems make use of closed-form solutions which have been developed over the last 75 years on the basis of quite restrictive assumptions. The idealizations which led to the formulation of the well-known Westergaard (1948) equations for a slab on dense liquid foundation, and of the less often quoted expressions for the corresponding slab on elastic solid subgrade case by Losberg (1960) and Ioannides (1988), treat a pavement system which:

(a) Considers a slab of infinite dimensions (no slab size effects);
(b) Consists only of one panel (no load transfer);
(c) Includes only one placed layer (no base or subbase);
(d) Employs a semi-infinite foundation (no rigid bottom);
(e) Is acted upon by a single tire-print (no multiple wheel loads); and

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(f) Experiences no curling or warping (flat slab, no temperature or moisture differential condition).

Each of these restrictions is violated in actual concrete pavement construction, which dictates that analytical considerations be adjusted or 'calibrated' before a reasonable engineering design can be made. In particular, the treatment of the concrete pavement as incorporating only one placed layer, viz. the PCC slab, has been one of the most pervasive obstacles in the effort to arrive at a mechanistic design which would permit comparisons with alternative designs involving asphalt concrete. The inability of conventional plate theory solutions to accommodate multiple placed layers is often cited as one of the primary reasons calling for its abandonment in favor of a "unified" analysis and design procedure based on layered elastic theory (Rice, 1989).

Use of layered elastic theory in addressing the single placed layer (SPL) limitation of conventional plate theory solutions is not a new suggestion. In fact, it is the oldest of at least three main approaches to the problem posed by bases underneath concrete pavement slabs. Even before the development of computer codes allowing the analysis of multi-layered axisymmetric pavement systems, layered elastic theory was suggested by Odemark (1949) -in the form of his celebrated Method of Equivalent Thicknesses- as a theoretically sound methodology for extending -rather than replacing- plate theory applications. The reason for not calling for the outright elimination of plate theory as an analytical tool for concrete pavement systems was a recognition by early investigators of the reciprocal inability of
layered elastic theory to consider the all important phenomena pertaining to the edges and corners of concrete slabs.

With the advent of sophisticated finite element codes, a second approach to the SPL problem emerged. This is exemplified in the treatment of concrete pavement systems as a two-layered composite plate resting on an elastic foundation, implemented into computer program ILLI-SLAB in the late 1970s by Tabatabaie, et al. (1979). Although treating both placed layers (i.e. slab and base) as plates does address the noted SPL shortcoming -particularly in the case of cement treated (stiff) bases- predictions on the basis of plate theory alone are often found to incorporate a significant error, arising from the neglected compression experienced by the two layers (especially when softer, unbound bases are employed).

The most predominant means for accounting for the presence of a base, however, has been by increasing the value of the subgrade modulus, \( k \). Thus, in contrast to Tabatabaie's formulation which considers the base as a structural element reinforcing the other placed layer, viz. the PCC slab, the more conventional approach has been to regard the base as contributing exclusively to the stiffness of the subgrade. It appears that the popularity of this approach is due more to the practical expediency and ease of solution it offers, rather than to its theoretical merits. A literature survey conducted at the beginning of the investigation reported in this Paper has identified at least twelve different ways of 'bumping the \( k \)-value,' or defining a composite or 'top-of-the-base' subgrade modulus. According to a review of current methods for determining the composite modulus of subgrade
reaction conducted by Uzan and Witczak (1985), "the equivalent $k_{comp}$-values for granular bases [obtained by different methods] are essentially the same," but for "stabilized materials... the k-values can vary within a factor of two."

Of the three approaches to dealing with the SPL assumption outlined above, the process of increasing the value of the subgrade modulus depending on the type and thickness of the base is the least attractive from a theoretical viewpoint. Its origins may be traced to tests conducted in the 1950s by the Portland Cement Association (PCA) and the Corps of Engineers (Yoder, 1959). At that time, the 'bump the k-value' approach appeared as a minor extrapolation of Methods 2 and 3 described by Teller and Sutherland (1935) for the determination of the subgrade modulus. Recall, however, that both of these methods (namely the volumetric approach and the backcalculation approach) aimed at defining a property of the natural subgrade, just as was the Plate Load Test (Teller and Sutherland's Method 1) rather than the property of an 'equivalent' supporting medium. In fact, it is precisely the development of computerized backcalculation procedures based on matching theoretical and observed deflection basins (by determining the area or volume of the basin) that has revealed the extent of errors that may be committed through the use of composite, top-of-the-base k-values. Such values are quite often much higher than those reported in earlier literature (Weissmann, et al., 1990), so much so that the definition of the medium they purport to describe as a dense liquid is brought into question.
This Paper offers a theoretically sound yet practical solution to the problem posed by the SPL assumption. This is in the form of simple equations that may easily be implemented on a personal computer or hand-held calculator and which may be used to calculate with sufficient accuracy the maximum responses in concrete pavement systems incorporating a base layer. As such, the formulae presented are extensions of well-known available closed-form solutions, obtained through the use of dimensional analysis concepts in interpreting a database of numerical results from more sophisticated computer codes. It is hoped that such solutions will eliminate the need to use empirical and theoretically questionable concepts such as that of the composite ‘top-of-the-base- subgrade modulus, thereby preventing any associated errors and miscalculations in the future.

SCOPE OF INVESTIGATION

The proposed analysis procedure for three-layer concrete pavements begins by considering the two placed layers as a composite plate and postulating that there exists an imaginary, homogeneous ‘effective’ plate resting on the same elastic foundation which deforms in the same manner as the real two-layer plate.

The purpose of the analysis presented below is:

(1) To verify the existence of the ‘effective’ plate, i.e. that it is possible to define its properties in terms of the corresponding properties of the two layers in the original composite plate;
(2) To obtain the elastic solution for the response of the 'effective' plate using available closed-form equations;
(3) To relate the 'effective' plate responses determined in this manner to the corresponding unknown composite plate responses; and
(4) To extend the applicability of the equations developed to the case of a three-layer concrete pavement system of any arbitrary stiffnesses, subject only to the assumption that one of the two placed layers is much stiffer than the foundation.

**CLOSED-FORM SOLUTION FOR THREE-LAYER SYSTEM WITH UNBONDED LAYERS**

(a) Plate Theory Solution

According to medium-thick plate theory (Timoshenko and Woinowsky-Krieger, 1959), when a flat plate of uniform cross-section is subjected to elastic bending, the following moment-curvature relationships apply, expressed in polar coordinates, \((r,\phi)\):

\[
M_r = -D \left[ \frac{\partial^2 w}{\partial r^2} + \mu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2} \right) \right]
\]

\[
M_\phi = -D \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2} + \mu \frac{\partial^2 w}{\partial r^2} \right)
\]

\[
M_{r\phi} = (1 - \mu) D \left( \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \phi} - \frac{1}{r^2} \frac{\partial w}{\partial \phi} \right)
\]
in which \( w(r, \phi) \) denotes the vertical displacement from the originally horizontal neutral axis of the plate. The flexural stiffness of the plate, \( D \), is defined by:

\[
D = \frac{E h^3}{12 (1 - \mu^2)}
\] (2)

where \( E \), \( \mu \) and \( h \) are the plate Young's modulus, Poisson ratio and thickness, respectively.

Equations (1) may be rewritten in a compact form as:

\[
\{M\} = -D \{L(\mu)\} \{w(r, \phi)\}
\] (3)

in which \( \{L(\mu)\} \) is a vector operator depending on the value of \( \mu \).

Consider now a composite plate, consisting of two dissimilar plate layers, resting on an elastic foundation, such as a dense liquid, or an elastic solid, etc. Assuming that during bending the two plate layers do not experience any separation, their respective deflection shapes will be identical, i.e.

\[
w_1(r, \phi) = w_2(r, \phi) = w_0(r, \phi)
\] (4)

Subscripts 1 and 2 denote here plate layers 1 and 2, respectively, in the original composite two-layer plate, while subscript \( e \) denotes an imaginary, homogeneous 'effective' plate resting on the same elastic

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foundation (see Fig. 1). The 'effective' plate is required to deform in the same manner as the real composite plate.

At this point, we introduce the assumption that the two plate layers in the composite plate act independently, i.e. that their interface is unbonded and free of shear stress. Application of Eqs. (3) and (4) to each of these plate layers yields the following moment expressions:

\[
\begin{align*}
\{M_1\} &= -D_1 \{L(\mu_1)\} \{w_e(r,\phi)\} \\
\{M_2\} &= -D_2 \{L(\mu_2)\} \{w_e(r,\phi)\}
\end{align*}
\]  

(5)  
and

(6)

The corresponding equation for the moment acting on the 'effective' plate is:

\[
\begin{align*}
\{M_e\} &= -D_e \{L(\mu_e)\} \{w_e(r,\phi)\}
\end{align*}
\]  

(7)

We also introduce the assumption that:

\[
\mu_1 = \mu_2 = \mu_e
\]  

(8)

and we note that the composite as well as the 'effective' plates are subjected to the same applied loads, experience the same deflections and, therefore, are acted upon by the same foundation reactions. Thus,
it is evident that:

\[ (M_T) = (M_1) + (M_2) = (M_e) \quad (9) \]

where \((M_T)\) denotes the total moment acting on the composite plate.

Equation (9) yields upon substitution from Eqs. (5) and (6):

\[ (M_e) = -(D_1 + D_2) (L(\mu_e)) [w_e(r,\theta)] \quad (10) \]

Comparison of Eq. (10) with Eq. (7) results in:

\[ D_e = D_1 + D_2 \quad (11) \]

Equation (11) verifies that the 'effective' plate postulated by Eq. (4) exists, and that its structural parameters can be defined in terms of the corresponding parameters of layers 1 and 2 in the original composite plate. Furthermore, it follows from Eqs. (7) and (11) that:

\[ (M_e) = (1 + \frac{D_2}{D_1}) (M_1) \quad (12) \]

Thus, the generalized stresses, \((\sigma_e)\), in the 'effective' plate are written in terms of the corresponding stresses, \((\sigma_1)\), in plate layer 1 as:

\[ (\sigma_e) = \frac{6}{h_e^2} \left( 1 + \frac{D_2}{D_1} \right) (M_1) \quad (13) \]
or

\[ \{ \sigma_e \} = \frac{h_1^2}{h_e^2} \left( 1 + \frac{D_2}{D_1} \right) \{ \sigma_1 \} \quad (14) \]

\[ \frac{h_1^2}{h_e^2} \left( \frac{E_1h_1^3 + E_2h_2^3}{E_1h_1^3} \right) \{ \sigma_1 \} \quad (15) \]

It follows from Eqs. (8) and (11) that:

\[ E_e h_e^3 = E_1h_1^3 + E_2h_2^3 \quad (16) \]

Substituting Eq. (16) into Eq. (15) leads to:

\[ \{ \sigma_e \} = \frac{h_1^2 E_e h_e^3}{h_e^2 E_1 h_1^3} \{ \sigma_1 \} \quad (17) \]

whence,

\[ \{ \sigma_e \} = \frac{h_e E_e}{h_1 E_1} \{ \sigma_1 \} \quad (18) \]

Setting \( E_e = E_1 \), this yields:

\[ \{ \sigma_e \} = \frac{h_e}{h_1} \{ \sigma_1 \} \quad (19) \]
and

$$
\sigma_1 = \frac{h_1}{h_e} \sigma_e
$$

(20)

Furthermore, the thickness of the 'effective' plate, \(h_e\), is obtained from Eq. (16) as:

$$
h_e = \sqrt[3]{h_1^3 + \frac{E_2}{E_1} h_2^3} = \sqrt[3]{h_1^3 + \frac{E_2}{E_1} h_2^3}
$$

(21)

Note that Eq. (20) implies, in particular, that the maximum bending stress, \(\sigma_1\), developing at the bottom of plate layer 1 in the composite plate may be obtained by multiplying the corresponding maximum bending stress, \(\sigma_e\), arising at the bottom of the imaginary, homogeneous 'effective' plate (of modulus \(E_e = E_1\)) by the thickness ratio \(h_1/h_e\), with \(h_e\) defined by Eq. (21). For example, considering the case of an elastic solid foundation, Eq. (20) implies that:

$$
\sigma_1 = \frac{h_1}{h_e} \sigma_e = \frac{h_1}{h_e} \sigma(h_e, E_1, E_S)
$$

(22)

The corresponding expression for a dense liquid foundation is:

$$
\sigma_1 = \frac{h_1}{h_e} \sigma_e = \frac{h_1}{h_e} \sigma(h_e, E_1, k)
$$

(22a)

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The notation $\sigma(h_i, E_j, F)$ denotes the maximum bending stress predicted by plate theory at the bottom of a plate of thickness $h_i$ and modulus $E_j$, resting on a subgrade characterized by generalized stiffness parameter $F$, i.e. Young's modulus, $E_g$, for an elastic solid foundation, or modulus of subgrade reaction, $k$, for a dense liquid foundation. Furthermore, Eq. (20) implies that the thickness ratio $(h_i/h_e)$ is the constant that relates the bending stress at any point $(r, \phi)$ in layer 1 of the composite plate to the bending stress arising at the corresponding point in the 'effective plate'. Having thus obtained $\sigma_1$, the maximum bending stress at the bottom of plate layer 2 may also be calculated using plate theory as follows, subject to the assumption of Eq. (8):

$$
\sigma_2 = \frac{E_2 h_2}{E_1 h_1} \cdot \sigma_1
$$

(23)

The value of $\sigma_e = \sigma(h_e, E_1, F)$ in Eqs. (22) and (22a) may be obtained using available closed-form solutions pertaining to the particular foundation type and loading condition of interest. An equation for the maximum bending stress arising at the bottom of a homogeneous infinite plate on an elastic solid foundation loaded by an interior load has been presented by Losberg (1960). More recently, Ioannides (1988) considered the edge and corner loading conditions for the same plate-foundation system, and provided simple formulae for the calculation of the maximum bending stress pertaining to these loading conditions, as well. The corresponding equations for a plate on dense liquid
foundation were given by Westergaard (1948), for all three fundamental loading conditions.

With respect to deflections, it is noted that Eq. (4) implies that the maximum deflection in the two-layer composite plate is equal to the maximum deflection experienced by the 'effective' plate. The latter may be calculated using the pertinent formulae given in the publications cited above. Similarly, the maximum subgrade stress under the composite plate is equal to the corresponding stress under the 'effective' plate.

(b) Elimination of Plate Theory Restrictions: Analytical Solution

To extend the applicability of the proposed approach to layers of any arbitrary stiffness -subject only to the assumption that one of the two layers is much stiffer than the foundation- responses calculated must be adjusted for the compression which occurs within the two layers of the original composite plate, and which is ignored by plate theory. To illustrate how such a corrective may be applied, the case of the maximum bending stress, \( \sigma_{1L} \), occurring at the bottom of layer 1 in a three-layer system of any arbitrary stiffnesses will be considered. A closed-form equation for this response may be written as:

\[
\sigma_{1L} = \sigma_1 + \theta \Delta \sigma
\]

(24)

where \( \sigma_1 \) is the corresponding stress according to plate theory given by Eq. (22), and \( \theta \Delta \sigma \) is a "correction increment". The contribution to
this increment of the compression of the second layer is usually of overriding importance. In a typical pavement system, the second layer generally has a much lower modulus than the first layer and may, therefore, be expected to diverge from plate behavior (no compression) more significantly than the first layer. For this reason, an expression for $\Delta \sigma$ accounting only for the compression in the second layer is derived first (i.e. $\theta = 1$). Considering the case of an elastic solid foundation, the following assumption is introduced at this point:

$$\frac{\partial (\Delta \sigma)}{\partial (E_s)} = 0$$  \hspace{1cm} (25)

i.e. that $\Delta \sigma$ (as well as the compression of the second layer) is largely insensitive to changes in the subgrade modulus, $E_s$. This assumption is a reasonable approximation for material moduli in the range of those typically encountered in concrete pavements, for which $E_1$ is much higher than $E_s$. If the assumption of Eq. (25) is accepted, the case may be considered in which $E_s = E_2$, which reduces the three-layer system to a two-layer system. Then, for this case:

$$\Delta \sigma(h_1, h_2, E_1, E_2, E_s) = \Delta \sigma(h_1, h_2, E_1, E_2)$$  \hspace{1cm} (26)

in which the parameters listed in parentheses define the properties of the layered system considered in determining $\Delta \sigma$. By referring to Eqs. (24) and (22), the following expression may be written:

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\[ \Delta \sigma(h_1, h_2, E_1, E_2, E_2) = \sigma_{IL}(h_1, h_2, E_1, E_2, E_2) - \frac{h_1}{h_e} \sigma(h_e, E_1, E_2) \quad (27) \]

If \( E_1 \gg E_2 \), \( \sigma_{IL} \) may be evaluated according to plate theory, or:

\[ \sigma_{IL}(h_1, h_2, E_1, E_2, E_2) = \sigma(h_1, E_1, E_2) \quad (28) \]

It is noted that in writing this Equation, the effect of an unbonded surface at depth \( h_2 \) into the elastic half-space of modulus \( E_2 \) is assumed to be negligible. Thus,

\[ \Delta \sigma(h_1, h_2, E_1, E_2, E_2) = \sigma(h_1, E_1, E_2) - \frac{h_1}{h_e} \sigma(h_e, E_1, E_2) \quad (29) \]

i.e. the correction increment \( \Delta \sigma \) may be calculated as the difference between two stresses each of which is evaluated using available closed-form solutions, such as those by Losberg (1960) or Ioannides (1988), for the parameters indicated by Eq. (29).

The value of \( \Delta \sigma \) obtained as explained above accounts only for the compression of the second layer, i.e. applies when \( E_2 \ll E_1 \). This would be the case, for example, of a Portland cement concrete (PCC) slab placed on a subbase. As \( E_2 \) tends toward \( E_1 \), Eq. (28) becomes increasingly inaccurate. Noting that for such pavements plate theory would apply without the need for corrections (since in this case both \( E_1 \) and \( E_2 \) are much higher than \( E_s \)), the correction increment should tend to zero.. In addition, when \( E_2 > E_1 \), the correction increment must
be negative, reflecting the effect of the compression in the first layer. This corresponds to the case, for example, of an asphalt concrete overlay on a PCC slab. For these reasons, therefore, the value of $\Delta \sigma$ obtained above is multiplied by a factor, $\theta$. Considering the interior loading condition, the following formula was developed for $\theta$ on the basis of comparisons of the proposed closed-form solution to the results of several three-layer runs of the BISAR computer program (Peutz, et al., 1968):

$$\theta = 1 - \exp \left( \frac{1}{3} \left[ 1 - \frac{E_1}{E_2} \right] \right)$$

(30)

Thus, substituting into Eq. (24) $\sigma_1$ from Eq. (22) and $\Delta \sigma$ from Eq. (29), the general solution for the maximum bending stress, $\sigma_{1L}$, arising at the bottom of the upper layer in an arbitrary three-layer system may be written as:

$$\sigma_{1L} = \frac{h_1}{h_e} \left\{ \sigma(h_e, E_1, E_2) + \theta \left( \sigma(h_1, E_1, E_2) \right) - \frac{h_1}{h_e} \sigma(h_e, E_1, E_2) \right\}$$

(31)

with $h_e$ as defined by Eq. (21) and $\theta$ as given by Eq. (30). Each of the three bending stresses $\sigma(h_i, E_j, E_k)$ in Eq. (31) may be calculated using Losberg's formula for the interior load-elastic foundation case. The accuracy of this formula has been improved during this study by the inclusion of the third order term, as follows:
\[ \sigma = \frac{-6 \ P \ (1 + \mu)}{h_i^2} \left[ -0.0490143 + 0.0795775 \ ln \left( \frac{a}{l_e} \right) \right. \\
\left. - 0.0120281 \left( \frac{a}{l_e} \right)^2 + 0.00353678 \left( \frac{a}{l_e} \right)^3 \right] \]

(32)

where:

\[ l_e = \sqrt[3]{\frac{E_j h_i^3 (1 - \mu_s^2)}{6 E_s (1 - \mu_j^2)}} \]

(33)

\( \mu_j \) and \( \mu_s \): Poisson ratios for the plate and foundation, respectively;

\( P \): total applied load; and

\( a \): radius of applied load.

The proposed procedure for calculating \( \sigma_{IL} \) is well suited for incorporation into a personal computer spreadsheet. Its applicability was verified by comparison of predicted values to the corresponding maximum bending stresses obtained by numerous three-layer solutions using computer program BISAR. Figure 2 illustrates clearly the excellent agreement obtained.

(c) Elimination of Plate Theory Restrictions: Graphical Solution

It is often desirable to assess the effect on the maximum bending stress of the introduction of a subbase under a PCC slab. Equation (31) is very well suited for this purpose. An alternative graphical
solution was also developed in this study. The derivation proceeds from Eq. (22) which may be re-written as:

\[ \sigma_1 = 6 \frac{M_e}{\eta_e^2} \]  \hspace{1cm} (34)

where:

\[ \eta_e^2 = \frac{h_e^3}{h_1} = h_1^2 + h_2^2 \left[ \frac{E_2}{E_1 h_1} \right] \]  \hspace{1cm} (35)

Noting that as \( \eta_e^2 \) tends to \( h_1^2 \), \( M_e \) tends to \( M_1 \) and \( \sigma_1 \) tends to \( \sigma(h_1, E_1, F) \) -the latter being the plate theory prediction for the maximum bending stress in the PCC slab resting directly on the subgrade- it may be expected that the stress ratio \( \frac{\sigma_1}{\sigma(h_1, E_1, F)} \) diverges from unity as the ratio \( \frac{\eta_e^2}{h_1^2} \) decreases. Salsilli (1991) considered the results of plate theory for a small factorial of dense liquid-edge loading cases, as implemented in the three-layer option in ILLI-SLAB, and demonstrated that the relationship between the two parameters defined above shows little sensitivity to the dimensionless load size ratio \( (a/l) \). He also provided the following best-fit equation for its description:
\[
\frac{\sigma_1}{\sigma(h_1, E_1, k)} = 0.0477629 + 0.265264 \left( \frac{a}{\ell} \right) + 0.953195 \left( \frac{\eta_e}{h_1} \right)^2
\]

(36)

\[-0.26083 \left( \frac{a}{\ell} \right) \left( \frac{\eta_e}{h_1} \right)^2\]

In this expression \( \ell \) denotes the radius of relative stiffness of the slab-dense liquid system \((h_1, E_1, k)\) which is defined by:

\[
\ell = \left( \frac{E_1 h_1^3}{12 (1-\mu_1^2) k} \right)^{1/4}
\]

(37)

Although Salsilli (1991) developed Eq. (36) with values for \( \sigma(h_1, E_1, k) \) obtained using the one-layer option in ILLI-SLAB, the value predicted by the Westergaard (1948) edge loading equation may be substituted in routine application of the formula.

Salsilli (1991) also suggested an alternative method for calculating the maximum bending stress, \( \sigma_1 \), by considering a slab of thickness \( \eta_e \) and modulus \( E_1 \) resting on the same dense liquid foundation and using directly Westergaard's equation, i.e.:

\[
\sigma_1 = \sigma(\eta_e, E_1, k)
\]

(38)

but provided no theoretical justification for his proposal. Comparison with Eq. (22a) shows that Eq. (38) is a good approximation when the dimensionless stress is not very sensitive to the \((a/\ell)\) ratio.
A drastically different picture is obtained when the compressions in the two man-made layers are accounted for. Computer program BISAR was executed a number of times so that the relationship between the two ratios, \( \sigma_{1L}/\sigma(h_1, E_1, E_2) \) and \( \eta_e^2/h_1^2 \), might be defined without any plate theory restrictions. Interpretation of these numerical results on the basis of the principles of dimensional analysis showed that for a wide range of practical applied load radius values, the relationship between the two ratios could be defined uniquely for each value of \((E_1/E_2)\). Thus, the chart in Fig. 3 was prepared. This can be used to obtain a "reduction factor," which when multiplied by the available closed-form slab-on-grade solution for \( \sigma(h_1, E_1, E_2) \) given by Losberg (1960) provides an estimate for the maximum bending stress at the bottom of the top layer in a three-layer system. The applicability of this chart was verified by a large number of additional BISAR test runs not included in the derivation of Fig. 3. The comparison presented in Fig. 4 confirms the accuracy of the proposed graphical solution.

CLOSED-FORM SOLUTION FOR THREE-LAYER SYSTEM WITH BONDED LAYERS

(a) Analytical Solution

The closed-form solution derived above for unbonded layers may be applied to the case of bonded layers as well, with relatively few modifications. The most significant of change is in the definition of the 'effective' thickness, \( h_e \). Recall, that for unbonded layers, \( h_e \) was defined using the condition of equality between flexural stiffnesses of the original composite two-layer plate and of the
imaginary, homogeneous 'effective' plate. Equation (16), however, applies only to unbonded layers. In the case of bonded layers, the flexural stiffness of the original composite plate may be determined using the parallel axes theorem. This results in the following alternative condition to Eq. (16):

\[
\frac{E_e h_e^3}{12} = \frac{E_1 h_1^3}{12} + E_1 h_1 \left[ x - \frac{h_1}{2} \right]^2 + \frac{E_2 h_2^3}{12} + E_2 h_2 \left[ h_1 - x + \frac{h_2}{2} \right]^2
\]

Equation (39) assumes that the neutral axis of the composite system lies within layer 1, at a distance \(x\) from the top of layer 1 (see Fig. 5), but the same expression is obtained if the neutral axis is assumed to lie within layer 2 (\(x\) is still measured from the top of layer 1). As done for the unbonded layers, it is assumed here that \(E_e = E_1\) and that \(\mu_e = \mu_1 = \mu_2\), which leads to the following expression for the thickness of the 'effective' plate for the case of bonded layers:

\[
h_e = \sqrt[3]{\left[ \frac{E_2}{E_1} \left( h_1^3 + \frac{E_2}{h_2^3} \right) \right] + 12 \left[ \frac{x - h_1}{2} \right]^2 + \frac{E_2}{E_1} \left[ h_1 - x + \frac{h_2}{2} \right]^2}
\]

The depth to the neutral axis, \(x\), is determined by considering the first moment of area of the original composite plate, as follows:

\[
x = \frac{E_1 h_1 \left( \frac{h_1}{2} + \frac{h_2}{2} \right) + E_2 h_2 (h_1 + \frac{h_2}{2})}{E_1 h_1 + E_2 h_2}
\]
Noting that the derivation of Equations (39) through (41) follow the same reasoning as used by Tabatabaie, et al. (1979), Eq. (40) may be rewritten as:

$$h_e = \sqrt{\frac{E_2}{E_1} h_1 F^3 + h_2 F^3}$$

(42)

where $h_1 F$ and $h_2 F$ are defined by:

$$h_1 F = \left[ h_1^3 + 12 \beta^2 h_1 \right]^{1/3}$$

(43)

$$h_2 F = \left[ h_2^3 + 12 \alpha^2 h_2 \right]^{1/3}$$

(44)

with:

$$\alpha = \left[ h_1 + \frac{h_2}{2} - x \right]$$

(45)

and

$$\beta = \left[ x - \frac{h_1}{2} \right] = \left[ \frac{h_1 + h_2}{2} \right] - \alpha$$

(46)

It is observed that Eq. (42) is identical to the corresponding Eq. (21) for $h_e$ for unbonded systems, the only substitution necessary being the introduction of the 'fictitious' thicknesses, $h_1 F$ and $h_2 F$, which are somewhat higher than the original thicknesses $h_1$ and $h_2$. It is also
clear that the flexural stiffness of the original composite two-layer bonded plate is equal to the stiffness of an unbonded two-layer plate in which the plate layers retain the moduli $E_1$ and $E_2$ but are assigned 'fictitious' thicknesses $h_{1F}$ and $h_{2F}$. The fact that $h_{1F} > h_1$ and that $h_{2F} > h_2$ counterbalances the effect of 'removing' the bond between the two plate layers.

A relationship between the bending stress at the bottom of the 'effective' plate, $\sigma_e$, and that acting at the bottom of layer 1 of the original composite two-layer plate, $\sigma_1$, is then sought. This is obtained with reference to the geometry of the stress distribution diagrams pertaining to the two systems, and recognizing their common slope above the neutral axis. As indicated in Fig. 6:

$$\frac{\sigma_e}{\sigma_1} = \frac{h_e}{2y}$$  \hspace{1cm} (47)

But:

$$y = (h_1 - x)$$  \hspace{1cm} (48)

whence:

$$\sigma_1 = \frac{2 \left( h_1 - x \right)}{\sigma_e} \frac{h_e}{h_e}$$  \hspace{1cm} (49)
This formula is similar in form to the corresponding Eq. (22) valid for unbonded systems, with the term $2(h_1-x)/h_e$ replacing $(h_1/h_e)$. The stress $\sigma_e$ may be evaluated using the available plate theory solutions pertaining to the loading condition and foundation type of interest. Since plate theory ignores the compression in each of the two layers, $\sigma_1$ should be corrected as indicated in Eq. (24). For the interior loading-elastic solid case, the necessary corrections are given by Eqs. (29) and (30). Note that the substitution of $(h_1/h_e)$ by $2(h_1-x)/h_e$ is also performed in Eq. (29). Thus, the following expression is obtained for $\sigma_{1L}$, corresponding to Eq. (31):

$$
\sigma_{1L} = \frac{2(h_1-x)}{h_e} \sigma(h_e,E_1,E_2) + \theta \left( \sigma(h_1,E_1,E_2) - \frac{2(h_1-x)}{h_e} \sigma(h_e,E_1,E_2) \right)
$$

with $h_e$ as defined by Eq. (42). A comparison between the stress predicted by this approach for the interior load-elastic foundation case and that determined using BISAR is shown in Fig. 7. Excellent agreement is observed, especially when the restriction $E_1/E_2 > 10$ is imposed (Fig. 8).

(b) Graphical Solution

An alternative graphical solution is also possible. Using $h_{1F}$ and $h_{2F}$ instead of $h_1$ and $h_2$, the ratio $[\eta_0F^2/h_{1F}^2]$ may be calculated from Eq. (35). Thus, the Chart in Fig. 3 may be used to calculate $\sigma_{1L}$ in terms of $\sigma(h_{1F},E_1,E_2)$. Figure 9 shows that the scatter between predicted and calculated (using BISAR) maximum bending stresses is
somewhat larger than was observed for the case of unbonded layers. The scatter is improved if the restriction $E_1/E_2 > 10$ is imposed on the cases considered, as shown in Fig. 10.

**IMPLICATIONS AND VERIFICATION OF PROPOSED APPROACH**

It is noted that in writing Eq. (31), no assumptions were made which would restrict it to the interior loading condition alone, with the exception of the fact that the axisymmetric program BISAR was used in the development of Eq. (30) for the factor $\theta$. Since, however, Eq. (30) is valid for both unbonded and bonded layers, it is reasonable to assume that it applies to edge and corner loading, as well. Thus, it is suggested Eq. (31) may be used in the analysis of three-layer concrete pavement systems under these loading conditions as well. For this purpose, the plate theory expressions given by Ioannides (1988) may be employed. Verification of this proposal would require the execution of a three-dimensional finite element code (Ioannides and Donnelly, 1988).

Furthermore, it may also be argued that it is possible to interpret the assumption of Eq. (25) as implying that the correction increment is independent of the nature of the foundation, as well. Thus, Eq. (31) may be applied to the case of a dense liquid foundation; the only necessary change being in the calculation of the plate theory stress using $\sigma(h_0,E_1,k)$ instead of $\sigma(h_0,E_1,E_0)$, where $k$ is the modulus of subgrade reaction. The correction increment, $\Delta\sigma$, is calculated using Losberg's equation as before. This proposal dispenses with the need to
define a 'top-of-the base' k-value, a procedure which often leads to erroneous conclusions (Ioannides, 1991). Although verification of this proposal for the edge and loading conditions would still require a three-dimensional finite element analysis, it is possible to examine the accuracy of the procedure for the dense liquid-interior loading case using a general purpose two-dimensional finite element code, such as FINITE (Lopez, 1977). This effort is continuing at this time.

For the determination of maximum deflections, it is suggested that this be taken as equal to the value computed using plate theory for any of the three fundamental loading conditions and for both the elastic solid and dense liquid foundations. These plate theory predictions should be corrected for the compression of the two placed layers. An effort in this direction is also under way.

Some evidence for the validity of these proposals is provided by a comparison of the maximum responses obtained with the plate theory solutions obtained using two-dimensional finite element program ILLI-SLAB (Tabatabaie and Barenberg, 1980). For this purpose, a database of 41 'typical' three-layer interior loading runs was developed. It was found that Eq. (22) yields the same stress as calculated using ILLI-SLAB, if μ₁=μ₂. If μ₁≠μ₂, the predicted stress is about 5% lower; a somewhat more accurate prediction is obtained in such cases using Eq. (38). Furthermore, it was verified that when the results of (uncorrected) plate theory are considered, plotting [σ₁/σ(hₑ,E₁,k)] versus (he/h₁)³ yields a unique curve. The curve developed in this study using σ₁ values obtained by ILLI-SLAB for interior loading is the same as the corresponding curve described by Eq. (36) which has been
developed by Salsilli (1991) for edge loading. This supports the assertion that the proposals above are valid for all three fundamental loading conditions for both the elastic solid and dense liquid foundations. Verification of the proposals for the elastic solid foundation is possible using the BOUSSINESQ option in ILLI-SLAB (Ioannides, et al., 1985).

The maximum deflection calculated using ILLI-SLAB was found to be the same as predicted by plate theory considering the 'effective plate' (thickness: \( h_e \); modulus: \( E_1 \)).

**CONCLUSION**

Analysis and design of concrete pavement systems has long been hampered by the restrictive assumption of available closed-form solutions of a slab resting directly on an elastic foundation. In reality -more often than not- concrete pavement slabs are placed on prepared bases, which are sometimes granular and sometimes bound. A number of approaches have been used in the last forty years to overcome the 'one placed layer' limitation. Most notable among these have been:

(a) Analyzing multi-layered concrete pavement systems using layered elastic analysis for axisymmetric conditions. Such applications include the Method of Equivalent Thicknesses (Odemark, 1948), and more recently computerized techniques, as implemented, for example, in program BISAR;
(b) Analyzing three-layer concrete pavement systems using a finite element program based exclusively on plate theory;

(c) Assigning an increased 'top-of-the-base' subgrade modulus purporting to reflect the structural contribution of the base layer, and analyzing three-layer concrete pavement systems using the available closed-form or numerical procedures.

Such approaches invariably suffer from considerable shortcomings and may in several cases lead to misguided conclusions. To remedy this situation, a practical approach for analyzing three-layer concrete pavement systems is presented in this paper, based on sound theoretical precepts and interpretation of numerical results using dimensional analysis. The proposed approach allows the calculation of the maximum responses in such systems - namely of deflection, bending stress, and subgrade stress - for all three fundamental loading conditions, and for both dense liquid and elastic solid foundations. It is shown that responses obtained on the basis of plate theory alone must be corrected for the compression experienced by the two placed layers. The implications of the proposed approach with respect to current analysis and design methodologies are far reaching. Research activities for its full verification and refinement are continuing at this time.

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REFERENCES


FIGURE 1 (a) Original Composite Two-Layer Plate on Elastic Foundation; (b) 'Effective' Homogeneous Plate on Elastic Foundation
FIGURE 2 Validation of Proposed Closed-Form Procedure for Maximum Bending Stress in Layer 1 Under Interior Loading (Elastic Solid Foundation)
$N = 137$

FIGURE 4 Validation of Proposed Graphical Procedure for Maximum Bending Stress in Layer 1 Under Interior Loading (Elastic Solid Foundation)
FIGURE 5 Location of Composite Neutral Axis for Bonded Layers
(a) Original Composite Two-Layer Plate on Elastic Foundation;
(b) 'Effective' Homogeneous Plate on Elastic Foundation
FIGURE 6 Stress Distribution in Bonded Plate-Layer System
(a) Original Composite Two-Layer Plate on Elastic Foundation;
(b) 'Effective' Homogeneous Plate on Elastic Foundation
FIGURE 7 Validation of Proposed Closed-Form Procedure for Max. Bending Stress in Layer 1 Under Interior Loading (Elastic Solid Foundation; Bonded Layers: All Data)
FIGURE 8: Validation of Proposed Closed-Form Procedure for Maximum Bending Stress in Layer 1 Under Interior Loading (Elastic Solid Foundation; Bonded Layers; Selected Data)

- $N = 52$
- $E_1 / E_2 > 10$
- $\sigma_{3-L} > 50 \text{ psi}$

Bending Stress BISAR (psi)

Bending Stress Closed-Form Procedure (psi)
FIGURE 9 Validation of Proposed Graphical Procedure for
Maximum Bending Stress in Layer 1 Under Interior Loading
(Elastic Solid Foundation; Bonded Layers: All Data)