10. Internal Waves

A major type of motion within a lake body is that of internal wave motion. We are most familiar with wave motion on the surface of a lake. One unique characteristic of stratified systems is that wave motion can also be present on any isopicnal surface (surface of constant density). In fact, the surface of the lake is itself an isopicnal surface in a stratified system containing the lake-atmosphere interface. In a similar way to surface waves, all lines of constant density within the body of a stratified lake may have periodic wave motion. In systems with continuous stratification these waves are called internal waves. In systems with step stratification, as in the water-atmosphere system, these waves are called interfacial waves or surface waves.

In this chapter we begin by considering interfacial waves in detail since they are easier to study mathematically and since they still remain a good model of the density structure in many natural environments (e.g. epilimnion-hypolimnion stratification in lakes, reservoirs, estuaries, and oceans). We first present the analysis for a bounded domain (rigid-lid approximation) and generalizations to a series of special cases. We continue by removing the top boundary in order to consider the effects of a free surface. In the final section the results for interfacial waves are extended to continuous stratification.

For further reading on internal and interfacial waves see Fischer et al. (1979), Wetzel (1983), Acheson (1990) and Buick (1997).

10.1 Bounded interfacial waves

Consider the stratified, two-layer, bounded system depicted in Figure 10.1. For our analysis we will make the following simplifying assumptions:

1. The two fluids are immiscible.
2. Both the upper and lower boundaries are rigid.
3. Horizontal motion is in only one direction.
4. Inviscid analysis is applicable (i.e. the viscosity approaches zero, implying the Reynolds number Re is large).
5. The flow within each layer is irrotational.
6. The interfacial waves are small amplitude linear waves (this assumption is elaborated further in the discussion of the boundary conditions).

Our first step is to derive the governing equations. Here, we will follow the traditional approach (see, e.g. Acheson (1990), Buick (1997), and Drazin (2002)). For this approach we define

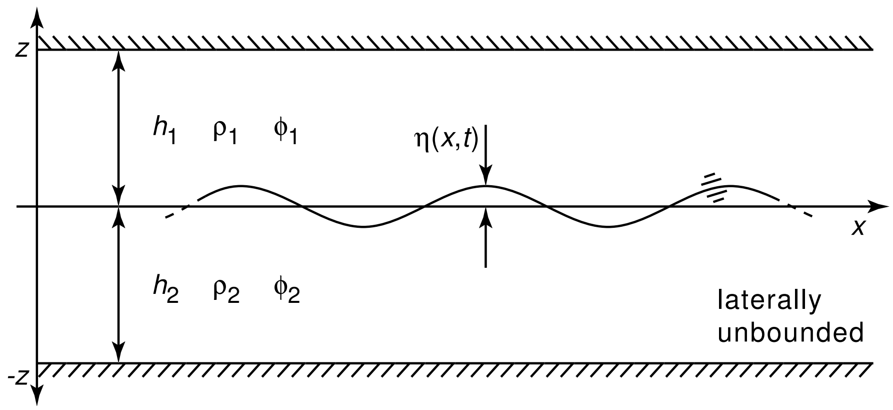

Fig. 10.1. Definition sketch for bounded interfacial waves.

a velocity potential, ϕ, such that

$$u = \frac{\partial \phi}{\partial x}; \ w = \frac{\partial \phi}{\partial z} \tag{10.1}$$

where u is the horizontal velocity and w is the vertical velocity in the fluid. Substituting ϕ into the mass conservation equation for an incompressible fluid we obtain

$$\begin{aligned} \frac{\partial u_i}{\partial x_i} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0. \end{aligned} \tag{10.2}$$

The final result (10.2) is known as the Laplace equation (also written as $\nabla^2\phi = 0$), which you may have encountered previously in heat transfer and the analysis of electric and magnetic field potentials. Because we assume that each layer is irrotational, we can define independent velocity potentials in each layer; thus, our governing equations are

$$\nabla^2 \phi_1 = 0; \tag{10.3}$$

$$\nabla^2 \phi_2 = 0. \tag{10.4}$$

The next step is to specify the boundary and intial conditions. Since our governing equations are second-order in two spatial coordinates, we have four boundary conditions in each layer for a total of eight boundary conditions. We also must satisfy momentum conservation (applied later as an auxiliary condition at the interface), which introduces a first-order derivative in time. Hence, we also have one initial condition for each layer, for a total of two initial conditions.

Four of the boundary conditions (two in each layer) and both initial conditions are satisfied by so-called behavioral boundary conditions (see Boyd (1989) for a detailed description of such boundary conditions). The boundary condition statement is: *the solution must be periodic in space and time.* This condition is satisfied implicitly by selecting a periodic general solution.

Two more boundary conditions are obtained from the bounded domain: *the vertical velocity must vanish at the upper and lower bounds*. This condition provides one boundary condition for each layer, namely

$$\left.\frac{\partial \phi_1}{\partial z}\right|_{z=+h_1} = 0; \tag{10.5}$$

$$\left.\frac{\partial \phi_2}{\partial z}\right|_{z=-h_2} = 0. \tag{10.6}$$

The rigid boundary condition for the upper boundary is often used as an approximation for the free surface boundary condition. Such an approximation is called the *rigid lid approximation*.

The remaining two boundary conditions come from the kinematic condition at the interface: *fluid particles can only move tangentially to the fluid interface*. To express this condition mathematically, we must consider the interface between the two fluids in more detail. The interface location z_i is defined by the function

$$F = z_i - \eta(x,t) = 0 \tag{10.7}$$

where $\eta(x,t)$ is the interface disturbance (refer to Figure 10.1). The interface itself is also a streamline, so we can say $F = 0$ as we follow the fluid along the streamline of the interface. Writing this Lagrangian reference frame statement in the Eulerian reference frame gives

$$\frac{\partial F}{\partial t} + \boldsymbol{u} \cdot \nabla F = 0 \tag{10.8}$$

at $z = \eta$. To simplify this equation we write it in non-dimensional form using the transformation variables

$$x = \lambda x^* \qquad z = a z^*$$
$$u = U_0 u^* \qquad w = U_0 w^*$$
$$t = \tfrac{a}{U_0} t^*$$

where λ is the wave length, a is the wave amplitude, U_0 is a characteristic fluid velocity and the variables denoted by * are the new non-dimensional variables. Substituting, we obtain

$$\frac{\partial \eta}{\partial t} + \frac{a}{\lambda} u \frac{\partial \eta}{\partial x} = w \tag{10.9}$$

where we have dropped the stars to simplify the notation. For small-amplitude waves $a \ll \lambda$, and we can neglect the non-linear convective term. This method of approximation is called *linearization*. Further, we can apply the linearized condition at $z = 0$ instead of $z = \eta$. Hence, the kinematic boundary condition is

$$\frac{\partial \eta}{\partial t} = w \tag{10.10}$$

which says that the vertical velocity of the fluid at the interface must equal the velocity of the interface. Since we have vertical velocities in both layers, this condition can be applied twice giving us

$$\left.\frac{\partial \phi_1}{\partial z}\right|_{z=0} = \left.\frac{\partial \eta}{\partial t}\right|_{z=0}; \tag{10.11}$$

$$\left.\frac{\partial \phi_2}{\partial z}\right|_{z=0} = \left.\frac{\partial \eta}{\partial t}\right|_{z=0}. \tag{10.12}$$

Finally, we must supply the initial conditions by enforcing momentum conservation. This provides a dynamic boundary condition at the fluid interface: *the normal stress of the fluid must be continuous across the interface* (Drazin 2002). For an inviscid fluid, this means that the pressure is continuous at the interface. Taking the inviscid momentum equation in the vertical direction, we have

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{10.13}$$

where D/Dt is the material derivative operator. Substituting the velocity potential and linearizing the convective terms of the material derivative leaves

$$\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{10.14}$$

We can now integrate with respect to z to obtain

$$\rho\frac{\partial \phi}{\partial t} + p + \rho g z = C_0 \tag{10.15}$$

where C_0 is an integration constant. This equation is also known as the unsteady Bernoulli equation, and must be constant along a streamline. Selecting the interface of the fluids as the streamline and recognizing $p_1 = p_2$ at the interface, we have

$$\rho_1 \left.\frac{\partial \phi_1}{\partial t}\right|_{z=0} = \rho_2 \left.\frac{\partial \phi_2}{\partial t}\right|_{z=0} + g\eta(\rho_2 - \rho_1). \tag{10.16}$$

This equation specifies the relationship between the spatial and temporal periodicity in the behavioral boundary conditions. In other words, our behavioral boundary condition cannot have arbitrary periodicity in space and time.

It is now possible to solve (10.3) and (10.4) given the above boundary and initial conditions. Since we are interested in simple harmonic motion, we specify the interface motion as

$$\eta(x,t) = a\cos(kx - \omega t) \tag{10.17}$$

where k and ω are the interfacial wave number and frequency, respectively. Since we have linearized the equations, the system response must have the same periodicity; thus, using the separation of variables technique, we seek solutions of the form

$$\phi_j = Z_j(z)e^{i(kx-\omega t)} \tag{10.18}$$

where $j = 1, 2$ is the fluid layer and where we have written the periodicity in terms of a cosine function through the use of implied real-part operators (recall $\cos(\alpha x) = \mathrm{Re}(e^{i\alpha x}) = Re(\cos(\alpha x) + i\sin(\alpha x))$). Substituting (10.18) into the Laplace equation, we obtain the following ordinary differential equation for Z_j:

$$\frac{\partial^2 Z_j}{\partial z^2} - k^2 Z_j = 0. \tag{10.19}$$

The general solution of (10.19) is

$$Z_j(z) = C_j e^{kz} + D_j e^{-kz} \tag{10.20}$$

where k is positive by definition.

Substituting (10.18) into the boundary conditions on the fixed boundaries, we have

$$C_1 k e^{kh_1} - D_1 k e^{-kh_1} = 0; \tag{10.21}$$

$$C_2 k e^{-kh_2} - D_2 k e^{kh_2} = 0. \tag{10.22}$$

Substituting (10.17) and (10.18) into the kinematic boundary conditions at the interface, we have

$$C_1 - D_1 = \frac{-ia\omega}{k}; \tag{10.23}$$

$$C_2 - D_2 = \frac{-ia\omega}{k}. \tag{10.24}$$

Solving this system of four equations in four unknowns, we obtain

$$C_1 = \frac{-ia\omega e^{-2kh_1}}{k(e^{-2kh_1} - 1)} \tag{10.25}$$

$$C_2 = \frac{-ia\omega e^{2kh_2}}{k(e^{2kh_2} - 1)} \tag{10.26}$$

$$D_1 = \frac{-ia\omega}{k(e^{-2kh_1} - 1)} \tag{10.27}$$

$$D_2 = \frac{-ia\omega}{k(e^{2kh_2} - 1)}. \tag{10.28}$$

Substituting into (10.20) and showing the algebra, we have

$$\begin{aligned} Z_1 &= \frac{-ia\omega e^{-2kh_1+kz}}{k(e^{-2kh_1} - 1)} + \frac{-ia\omega e^{-kz}}{k(e^{-2kh_1} - 1)} \\ &= \frac{-ia\omega}{k} \frac{e^{-2kh_1+kz} + e^{-kz}}{e^{-2kh_1} - 1} \\ &= \frac{-ia\omega}{k} \frac{e^{-kh_1}(e^{-kh_1+kz} + e^{kh_1-kz})}{e^{-kh_1}(e^{-kh_1} - e^{kh_1})} \\ &= \frac{+ia\omega}{k} \frac{e^{k(z-h_1)} + e^{-k(z-h_1)}}{e^{kh_1} - e^{-kh_1}} \\ &= \frac{ia\omega \cosh(k(z-h_1))}{k \sinh(kh_1)} \end{aligned} \tag{10.29}$$

and, similarly,

$$Z_2 = \frac{-ia\omega \cosh(k(z+h_2))}{k \sinh(kh_2)}. \tag{10.30}$$

Thus, the velocity potentials become

$$\phi_1 = +\frac{ia\omega \cosh(k(z-h_1))}{k \sinh(kh_1)} \cos(kx - \omega t) \tag{10.31}$$

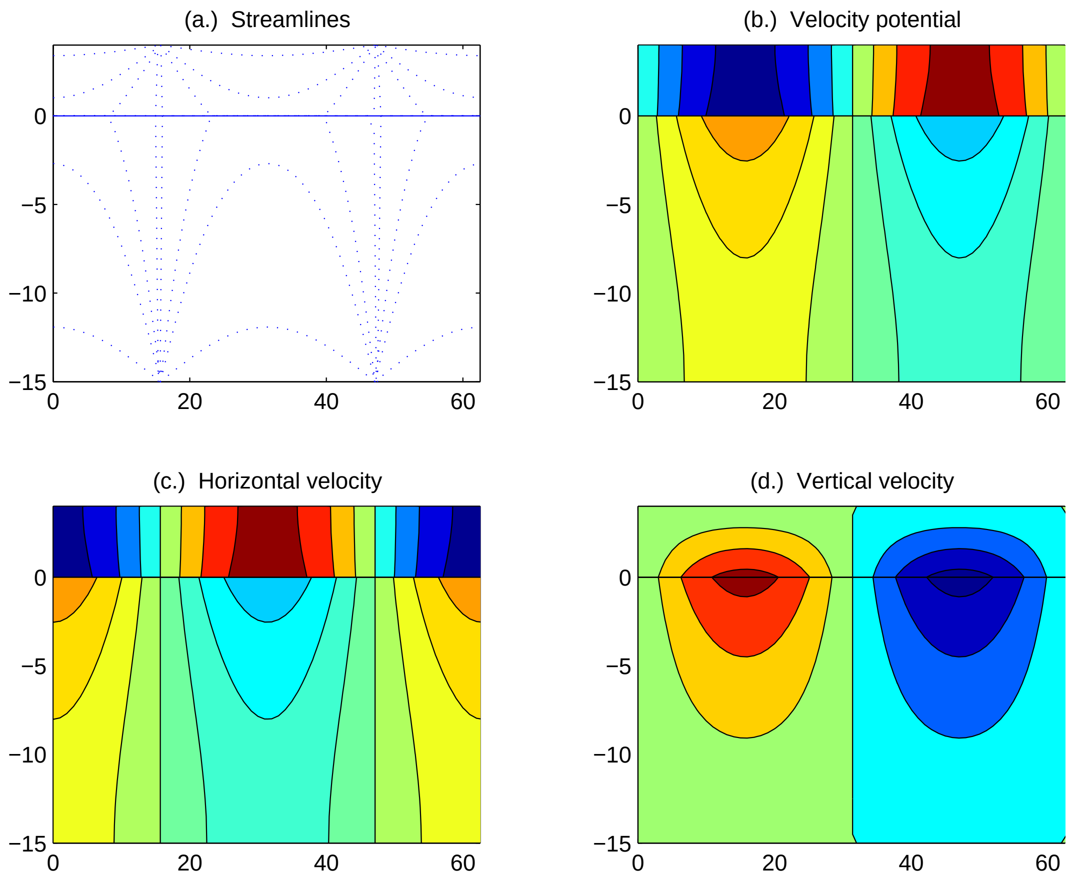

Fig. 10.2. Solutions for interfacial waves: (a.) shows the streamlines, (b.) shows the related velocity potential, (c.) shows the horizontal velocity component and (d.) shows the vertical velocity component.

$$\phi_2 = -\frac{ia\omega\cosh(k(z+h_2))}{k\sinh(kh_2)}\cos(kx-\omega t). \tag{10.32}$$

As mentioned above, the auxiliary dynamic boundary condition specifies the relationship between k and ω. Substituting ϕ_1, ϕ_2 and η into (10.16) gives:

$$\rho_1\frac{\omega^2}{k}\frac{1}{\tanh(kh_1)} + \rho_2\frac{\omega^2}{k}\frac{1}{\tanh(kh_2)} - g(\rho_2-\rho_1) = 0 \tag{10.33}$$

which is known as the *dispersion relation*. Thus, (10.31) through (10.33) together with (10.17) provide a complete solution for our bounded interfacial wave (including internal fluid velocities through the velocity potential).

Figure 10.2 plots our solution to the internal wave problem. Plot (a.) shows the streamlines (see the exercises at the end of this chapter); Plot (b.) plots the velocity potential. Plots (c.) and (d.) plot the horizontal and vertical velocity, respectively. Notice that the horizontal velocity is discontinuous at the interface, but that the vertical velocity is continuous at the interface and goes to zero at the edges. Thus, our boundary conditions were accurately reproduced in the solution.

10.2 Special cases

Having the general solution for interfacial waves in the bounded case, we now investigate the behavior of this solution for a range of special (limiting) cases.

10.2.1 Deep (unbounded) interfacial waves

We first consider deep waves in an unbounded domain ($h_1 \to h_2 \to \infty$). For the general solution (10.20) not to blow up at $\pm\infty$ we have $C_1 = D_2 = 0$. After applying the interface boundary conditions, we obtain

$$\phi_1 = \frac{ia\omega}{k} e^{-kz+i(kx-\omega t)} \tag{10.34}$$

$$\phi_2 = \frac{-ia\omega}{k} e^{kz+i(kx-\omega t)}; \tag{10.35}$$

thus, the hyperbolic functions used in the bounded case to force the vertical velocity to vanish at the boundaries have become simple exponential functions, $e^{\pm kz}$. Substituting $h_1 \to h_2 \to \infty$ into (10.33), the dispersion relation in deep water becomes

$$\omega_d^2 = \frac{gk(\rho_2 - \rho_1)}{\rho_1 + \rho_2}. \tag{10.36}$$

The phase speed of the wave is found by following a fixed point on the wave form as it propogates through time. To follow a fixed point we must have the total derivative of η remain constant, or

$$d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial t} dt = 0. \tag{10.37}$$

Substituting (10.17) for η and rearranging we have

$$\frac{d\eta}{dt} = c = \frac{\omega}{k} \tag{10.38}$$

where c is called the phase speed. From our simplified dispersion relation in deep water (10.36) the phase speed is

$$c_d = \pm\sqrt{\frac{g(\rho_2 - \rho_1)}{k(\rho_1 + \rho_2)}}. \tag{10.39}$$

Surface waves. The dispersion relation for surface water waves in deep water can be obtained by taking $\rho_1 = 0$. This gives the familiar deep-water results of

$$w_s = \pm\sqrt{gk} \tag{10.40}$$

$$c_s = \pm\sqrt{\frac{g}{k}}. \tag{10.41}$$

Now it is possible to see where the name dispersion relation comes from. For waves with longer wavelength, k is smaller and c is larger. Thus, long waves travel faster than short waves. If a spectrum of waves is generated in one region and permitted to propogate away, the long waves will move out ahead of the short waves and the wave field will sort itself (or disperse) according to wave length.

Boussinesq waves. Another limiting case is the Boussinesq wave, where $\rho_1 \approx \rho_2$. In this case, the denominator of the general solution can be simplified using an average density, ρ_0, to give

$$w_b = \pm\sqrt{\frac{g'k}{2}} \tag{10.42}$$

$$c_b = \pm\sqrt{\frac{g'}{2k}}. \tag{10.43}$$

where g' is the reduced gravity, $g(\rho_2 - \rho_1)/\rho_0$. Hence, the internal Boussinesq waves travel slower than surface waves due to the buoyant effect of g'.

10.2.2 Shallow internal waves.

Shallow water waves are defined by $kh \to 0$ such that $\tanh(kh) \to kh$. As a rule-of-thumb, this approximation is valid for $kh \leq 0.5$, or $\lambda \geq 4\pi h$. For this case the phase speed becomes

$$c_{sh} = \pm\sqrt{\frac{gh_1h_2(\rho_2 - \rho_1)}{\rho_1 h_2 + \rho_2 h_1}}. \tag{10.44}$$

Since c_{sh} does not depend on k, shallow water waves are not dispersive.

10.3 Effects of a free surface

We now relax the rigid lid boundary condition and consider the effects of a free surface. In the analysis so far, we applied the unsteady Bernoulli equation at the interface a obtain a dispersion relation that is second order in ω. By adding the free surface, we must apply this boundary condition a second time, and after working through the algebra, we obtain a dispersion relation of the form

$$\omega^4 = f(k, h_1, h_2, \Delta\rho/\rho). \tag{10.45}$$

For the rigid lid case, we obtained two solutions (i.e. $c = \pm\sqrt{gh}$) which actually correspond to one mode of behavior—waves travel with a single speed but in two directions. The four solutions given by equation (10.45), therefore, can be grouped into two different modes of behavior, each with waves traveling in two directions. These two modes of behavior have many names, but here we will distinguish them by their wave speeds and call them the "fast" mode and "slow" mode.

10.3.1 Fast mode

Assuming long, Boussinesq waves, the fast-mode solution to (10.45) simplifies to

$$\begin{aligned} c^2 &= g(h_1 + h_2) \\ &= gH. \end{aligned} \tag{10.46}$$

where H is the total water depth $h_1 + h_2$. Figure 10.3 shows this mode in the upper panel using another name for this mode: the external mode. This name arises from the fact that the

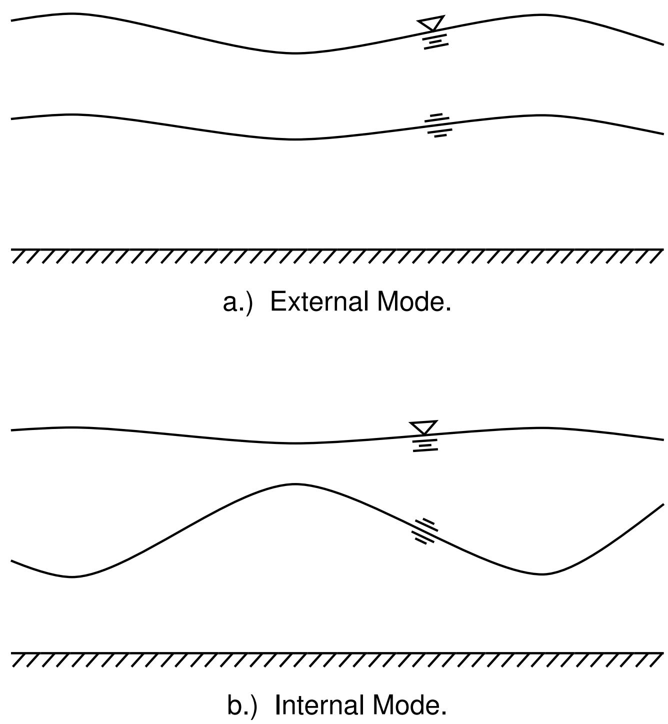

Fig. 10.3. Definition sketches for the two modes of behavior for the dispersion relation with a free surface.

dominant wave is the surface, or external, wave. Because this solution is independent of the density differences between layers, we say that the fast mode solution behaves as if the water body is unstratified.

Another property of this solution is that the pressure is constant along lines of constant density. Systems with this property are called barotropic; thus, this is also called the barotropic mode. Because the internal and surface waves move in phase, this can also be called the sinuous mode. Each of the terms used to name this mode draw out a different property of this solution, but all of these terms are referring to this single, fast mode of behavior.

10.3.2 Slow mode

The other solution to (10.45) for long, Boussinesq waves simplifies to

$$\begin{aligned} c^2 &= g' \frac{h_1 h_2}{h_1 + h_2} \\ &= g' \frac{h_1 h_2}{H}. \end{aligned} \tag{10.47}$$

Because the gravity term in (10.47) is g', which is small, the wave speeds for this mode are much slower than for the external mode, hence, the terms fast and slow modes. The lower panel in Figure 10.3 shows this mode, referred to by another name: the internal mode. Because the internal wave motion is much greater than the surface expression of this mode, this mode is also called the internal mode.

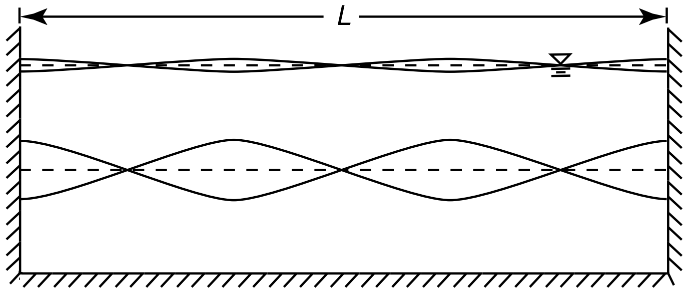

Fig. 10.4. Sketch of a standing wave for the internal wave mode.

In contrast to the fast mode, the pressure varies along lines of constant density. Systems with this kind of forcing are called baroclinic; hence, this mode is also called the baroclinic mode. Because the internal and surface waves are 180° out of phase, this mode can also be called the varicose mode.

As the figure also illustrates, the surface motion for the internal mode is very small in comparison to the internal wave motion. This behavior is what allows us to apply results obtained with the rigid lid approximation to systems with a free surface for the internal mode.

10.4 Effects of lateral boundaries: Standing waves

In natural water bodies, such as lakes and reservoirs, lateral boundaries are also present. Waves that encounter these boundaries are reflected, and the returning waves must be superposed with the initial wave train.

Under certain conditions, the reflected and progressive waves match in such a way that the waves appear to stand still—the waves move up and down, but do not appear to progress. Such waves are called standing waves. Figure 10.4 illustrates this situation for the internal, slow wave mode. The condition for standing waves is some integral number of half wave-lengths must fit within the bounded basin. Stated mathematically, we have

$$L = \frac{n\lambda}{2} \tag{10.48}$$

where λ is the wave length. The possible associated wave numbers are, then

$$\begin{aligned} k &= \frac{2\pi}{\lambda} \\ &= \frac{n\pi}{L} \end{aligned} \tag{10.49}$$

Since wave speed c is $c = \omega/k$, the associated wave speed and wave periods for standing waves are

$$c = \frac{2L}{nT} \tag{10.50}$$

and

$$T = \frac{2L}{nc}, \tag{10.51}$$

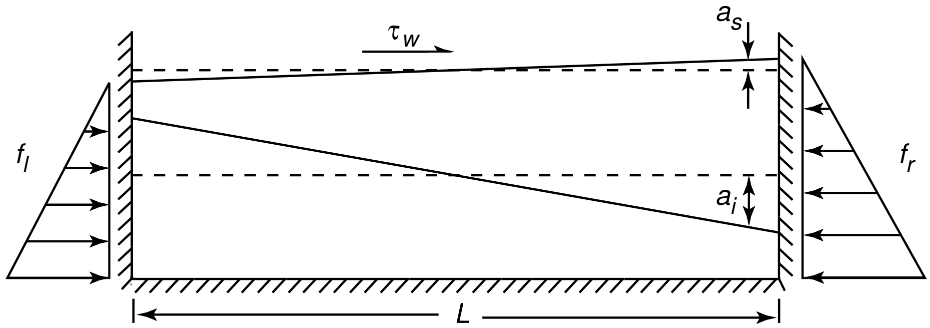

Fig. 10.5. Sketch of the set-up generated by the wind which becomes the initial condition for seiching motion.

respectively.

Substituting the results for the internal and external wave speeds given above into the standing wave period given in (10.51) gives

$$T_e = \frac{2L}{n\sqrt{gH}} \tag{10.52}$$

for the external mode and

$$T_i = \frac{2L}{n\sqrt{g'\frac{h_1 h_2}{H}}} \tag{10.53}$$

for the internal mode. Thus, we see again that the internal mode is much slower than the external mode.

10.4.1 Lake seiches

Seiches are basin-scale standing waves which can be either external or internal mode solutions. The word "seiche" comes from the French for waves and was first applied to basin-scale surface waves in Lake Geneva.

Seiches are generated by the wind. When the wind blows, water piles up on the downwind side. This generates circulating currents in the lake due to pressure imbalances. This condition while the wind is blowing is shown in Figure 10.5 and is called the wind set-up. When the wind stops, the surface moves back down and undergoes period wave-like motion. Because the set-up is like half a wave length, the waves generated are generally standing waves with $n = 1$.

To calculate the degree of set-up, consider a force balance on the lake. During the wind setup, the water body is held in the position shown in Figure 10.5, and the total external force acting in the x-direction must be zero. The forces are the hydrostatic force on the left F_l and right F_r and the force due to the wind on the surface $\tau_w A$, where A is the area over which tau_w is acting. Taking a unit width into the page, the force balance in the x-direction becomes:

$$\Sigma F_x = F_r - F_l - \tau_w L = 0 \tag{10.54}$$

where L is the length of lake along which the wind acts. To calculate F_l and F_r, we make the Boussinesq approximation and treat the water density as a constant rho. Then, the hydrostatic pressure distribution gives

$$F_l = \frac{1}{2}\rho g(H - a_s)(H - a_s) \tag{10.55}$$

$$F_r = \frac{1}{2}\rho g(H + a_s)(H + a_s) \tag{10.56}$$

where a_s is the setup height. Substituting into (10.54), gives

$$2\rho g a_s H - \tau_w L = 0$$

$$a_s = \frac{2\tau_w L}{\rho g H} \tag{10.57}$$

For order-of-magnitude estimates, τ_w may be estimated by $\tau_w \simeq 10^{-3}\rho_{\text{air}}U_{10}^2$, where U_{10} is the air velocity at 10 m height above the lake. Since the pressure on the water surface is everywhere zero, this set-up condition is called the barotropic forcing.

In response to this surface set-up, the interface sets-down in order to give a uniform pressure on the bottom. The pressure at the bottom on the down-wind side is

$$P_2 = \rho_1 g(h_1 + a_s + a_i) + \rho_2 g(h_2 - a_i) \tag{10.58}$$

and the pressure at the up-wind side on the bottom is

$$P_1 = \rho_1 g(h_1 - a_s - a_i) + \rho_2 g(h_2 + a_i). \tag{10.59}$$

Equating these two pressures gives

$$a_s = \frac{\Delta\rho}{\rho} a_i. \tag{10.60}$$

Thus we see that the surface set-up is much smaller than the internal adjustment since $\Delta\rho/\rho$ is small. Since the pressure on the interface is not a constant, the set-down response is called baroclinic adjustment. When the wind stops, this set-up and set-down situation becomes the initial condition for seiche motion. The seiche period and wave speed are computed from the dispersion relation using the wave number.

10.5 Internal waves in continuous stratification

Although many situations can be approximated as two-layered systems, density gradients in nature are always continuous, and we stop here to consider the effects of continuous stratification on wave propagation. In a lake, for instance, the waves at the interface can be approximated by the two-layer model; whereas, internal waves in the hypolimnion are more affected by the continuous stratification in the lower layer. Since the upper layer is mixed, internal waves do not form in the epilimnion.

There are two major differences between the analysis in continuous stratification and the analysis presented above. First, we cannot make the irrotational flow assumption in a continuously stratified medium; thus, we cannot define a velocity potential. Second, because the medium has vertical density variation, waves can travel horizontally along isopicnals and vertically along the density gradient. Figure 10.6 illustrates such a wave phenomenon. To describe the two-dimensional waves we require a two-dimensional wave number, namely

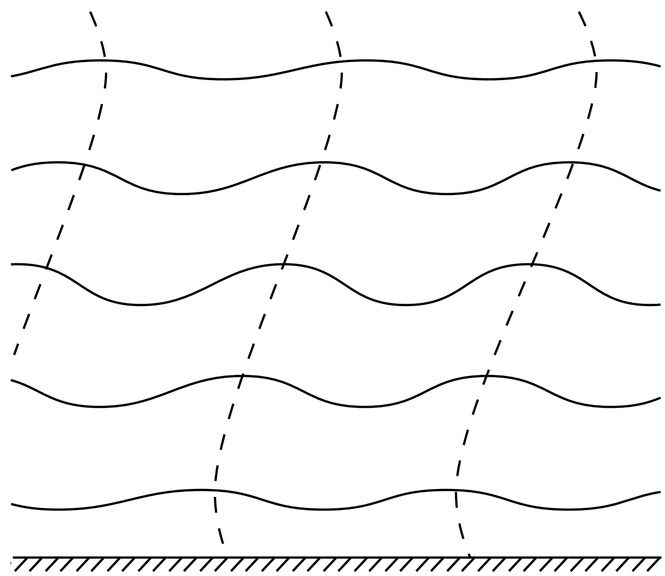

Fig. 10.6. Sketch of internal waves in continuous stratification. The solid lines show the horizontal motion and the dashed lines show the vertical motion.

$$\boldsymbol{k} = k\boldsymbol{\imath} + m\boldsymbol{\jmath} \tag{10.61}$$

where the horizontal wave length $\lambda_h = 2\pi/k$ and the vertical wave length $\lambda_v = 2\pi/m$.

We have similar governing equations to the two-layered case. For the conservation of mass of an incompressible fluid we have

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{10.62}$$

The momentum conservation equation is written for the linearized inviscid case using the Boussinesq approximation yielding

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0 \tag{10.63}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial z} + g' = 0. \tag{10.64}$$

This gives us three equations in the four unknowns u, w, ρ, and p'. The fourth equation comes from the linearized density conservation equation:

$$\frac{D\rho}{Dt} = 0$$

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho}{\partial z} = 0$$

$$\frac{\partial g'}{\partial t} - N^2 w = 0 \tag{10.65}$$

where we have multiplied the first equation by g/ρ to obtain the final result.

Solving this system of equations with the appropriate boundary conditions leads to a new dispersion relation given by

$$\omega^2 = N^2 \frac{k^2}{k^2 + m^2}. \tag{10.66}$$

Since the term $k^2/(k^2 + m^2)$ is less than or equal to one, the excitation frequency, ω, must be less than or equal to N. When $\omega > N$ no wave exists and the excitation is locally dissipated by mixing.

Now we can consider the two-dimensional propogation of the wave relative to the excitation vector. Consider horizontal excitation

$$\boldsymbol{\omega} = \omega \boldsymbol{\imath}. \tag{10.67}$$

The angle, θ, between the excitation and the wave number is given by the dot product $\boldsymbol{k} \cdot \boldsymbol{\omega}$ such that

$$\begin{aligned} \cos\theta &= \frac{\boldsymbol{k} \cdot \boldsymbol{\omega}}{|\boldsymbol{k}||\boldsymbol{\omega}|} \\ &= \sqrt{\frac{k^2}{k^2 + m^2}} \\ &= \frac{\omega}{N}. \end{aligned} \tag{10.68}$$

Here, we see again that for $\omega > N$ there cannot be a wave since the $\cos\theta$ does not exist.

10.5.1 Slow excitation

For slow excitation frequencies we have $\omega \ll N$ and $\cos\theta \approx 0$. In this case the wave propogation is perpendicular to the excitation direction. This can lead to the formation of absorption layers described below. The slow excitation caused by internal seiche currents interacting with the boundary, for instance, can generate wave trains that propogate into the lake body.

10.5.2 Eigenfrequency excitation

Forcing the system at the Eigenfrequency means $\omega = N$ and $\cos\theta \approx 1$. In this case the wave propogation is in the same direction as the excitation. This can lead to the formation of reflection layers discussed below.

10.5.3 Fast excitation

Fast excitation is for frequencies above the stratification frequency: $\omega > N$ and $\cos\theta$ does not exist. We pause here to consider why waves do not exist for this case. We know that the buoyancy frequency N is the oscillation frequency of the stratification. If we force the system faster than this frequency that means we are trying to get the system to swing faster than it can; hence, no waves form. This is like the child on a swing who is pumping his legs very fast. To get the swing to move he must pump at or below the Eigenfrequency of the swing pendulum.

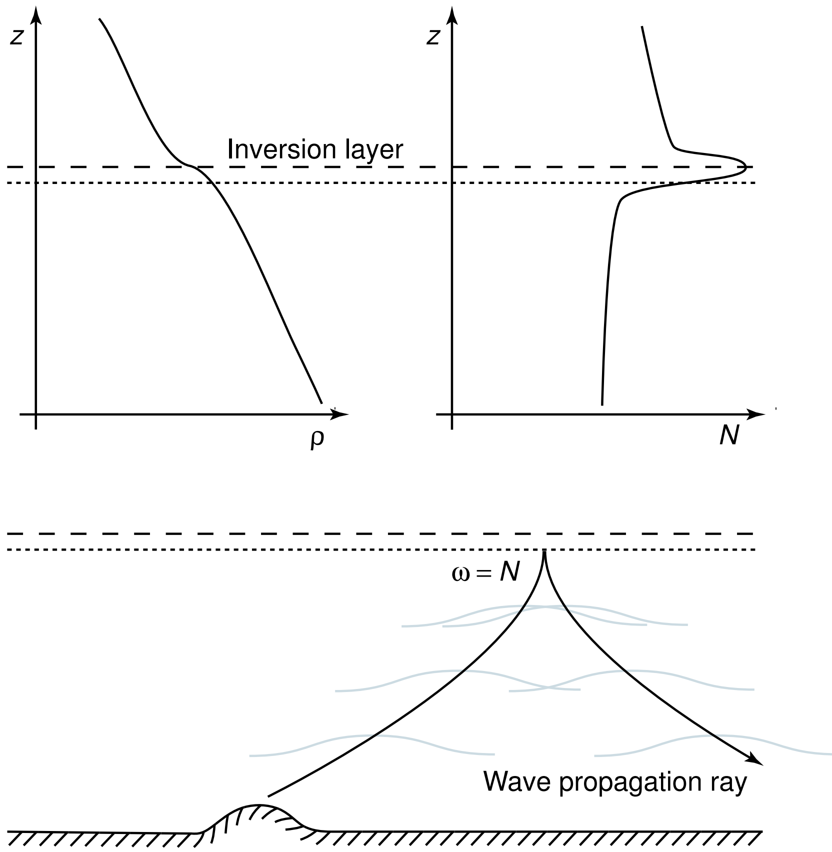

Fig. 10.7. Sketch of internal waves in the atmosphere. The upper two plots show the density and buoyancy frequency variation. The lower diagram shows the wave propagation ray for a given disturbance frequency ω, which depends on the wind speed, the mountain shape, and the stratification.

10.5.4 Waves in non-linear stratification

If the stratification is non-linear, then there is a variation of N over the depth. Consider, then, a given excitation frequency ω. If we apply this excitation in a region where $\omega > N$, then no waves will form. If we apply this excitation in a region where $\omega < N$, waves form and propogate at an angle θ to the excitation vector given by (10.68). As these waves propogate, they encounter fluid with different N and the wave rays bend. When the waves encounter a layer where $\omega = N$ they are reflected back to where they came from. When waves move into a layer where $\omega \ll N$ they move parallel to this layer and become trapped. Figure 10.7 illustrates the situation for atmospheric internal waves above mountains in an inversion layer.

Summary

This chapter introduced wave motion in stratified water bodies. The simple case of two immiscible layers bounded above and below was treated first in detail. The result of the analysis is the dispersion relation, which gives the relationship between wave speed and wave number.

Limiting cases were also investigated that show simplified versions of the dispersion relation. The boundary at the top was then removed and the new dispersion relation introduced. Because internal wave motion has only a very small surface expression, the bounded results are shown to agree well with the unbounded solution. Finally, the effects of continuous stratification were introduced, in particular, the fact that waves travel in two dimensions.

Exercises

10.1 Dispersion relations. Show that the shallow water wave approximation leads to $c = \pm\sqrt{gH}$ for surface waves and $c = \pm\sqrt{g' h_1 h_2/(h_1 + h_2)}$ for Boussinesq waves.

10.2 Streamlines. Calculate the stream function, ψ, for the bounded internal wave case and show that the streamlines are defined by

$$z_1 = \frac{1}{k}\sinh^{-1}\left[-\frac{Sk}{a\omega}\sinh(kh_1)e^{-i(kx-\omega t)}\right] + h_1 \tag{10.69}$$

$$z_2 = \frac{1}{k}\sinh^{-1}\left[+\frac{Sk}{a\omega}\sinh(kh_2)e^{-i(kx-\omega t)}\right] - h_2 \tag{10.70}$$

where S is constant along a streamline.

10.3 Lake seiches. Calculate the wave period for the first and second internal and external modes for a lake of length $L = 3$ km, depth $H = 30$ m, mixed layer depth $h_1 = 3$ m, upper layer temperature of $T_1 = 22$ °C, and bottom layer temperature of $T_2 = 10$ °C. If each wave is damped after 7 wave periods, how long does the external and internal wave last? What is the Eigenfrequency of the first internal wave mode?

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