8. Concepts, Definitions, and Governing Equations

Stratified fluids are ubiquitous in nature, present in almost any heterogeneous fluid body. Examples include thermal stratification of reservoirs and oceans, salinity stratification in estuaries, rivers, groundwater reservoirs, and oceans, heterogeneous mixtures in industrial, food, and manufacturing processing, density stratification of the atmosphere, and uncountable similar examples. In the presence of gravity, these density differences have a dramatic impact on the dynamics and mixing of heterogeneous fluids. For example, thermal stratification in reservoirs can reduce the vertical mixing of oxygen to the point that bottom water becomes anoxic through the action of biological processes. Preventing, predicting, and solving such a reservoir problem, though dependent on other limnological factors, requires an understanding of the dynamics of stratified fluids.

This chapter introduces the terminology and dynamic equations that will be used in the remainder of this book to introduce several classical problems in stratified fluids. These problems will include reservoir mixing, internal waves, shear flow instability, internal hydraulics, turbulence, and jets, plumes and wakes. One of the goals of this chapter is to translate key phrases in stratified fluid mechanics into mathematical operations that will then be implemented throughout this text.

8.1 Concepts and Definitions

A stratified fluid is one in which the fluid density $\rho$ is a function of space $(x, y, z)$ and (sometimes) time $t$

$$\rho = f(x, y, z, t) \equiv \left[ \frac{M}{L^3} \right]; \quad (8.1)$$

$\rho$ has units mass $M$ per volume $L^3$. We will normally define $z$ as positive vertically upward. The term stratification refers to the variation in the density field.

Density differences arise through many sources. These can include differences in temperature, dissolved phases (solids, liquids, and gases), suspended solids, and pressure differences. Some of these causes are due to the properties of the fluid itself (as in the case of temperature or pressure differences), but many density differences result from the mixture of two or more solutions or phases. Mixtures generally fall into four main categories:

1. Miscible: fluids that can mix completely with each other to form a homogeneous mixture (e.g., alcohol and water).
2. Immiscible: fluids that do not readily mix with each other and will readily separate after mixing (e.g. cooking oil in water),

3. Emulsible: fluids which act as immiscible until rigorously mixed. The mixing action causes one of the fluids to become trapped in a matrix of the other, forming a so-called emulsion that can have very different properties from the parent fluids. Emulsions can either be stable (remaining in the emulsion phase for long times) or unstable (slowly separating into the original two fluids). Examples include emulsifiable crude oils mixed with water; the emulsified product is often gelatinous, such as vaseline.

4. Multiphase: mixtures containing more than one phase state (e.g. air bubbles in water or sediment loads in rivers). Immiscible fluid mixtures are often also referred to as multiphase since their behavior is analogous to multiphase behavior.

For a miscible fluid mixture, the fluid density can be represented as a function of its temperature, pressure, and concentration of dissolved components. Such an equation is called the equation of state of the fluid.

An important equation of state that we will use is that for seawater. The density of seawater is often reported in \( \sigma_t \) units, defined as

\[
\sigma_t = \rho_{sw} - 1000 \quad (8.2)
\]

where \( \rho_{sw} \) is the density of seawater in kg/m\(^3\). Figure 8.1 plots \( \sigma_t \) as a function of salinity (a measure of the amount of dissolved salts in the seawater) and temperature. Conveniently, oceanographers have shown that the composition (ratio of various component salts to one another) of ocean water does not vary much around the globe. As a result, the density of seawater can be written as a function of the practical salinity: the total mass of dissolved salts per mass of water. The UNESCO equation of state for standard mean ocean water at Standard Temperature and Pressure (STP) is

\[
\rho_{sw}(S,T,0) = \rho_w + (8.24493 \times 10^{-1} - 4.0899 \times 10^{-3}T \\
+ 7.6438 \times 10^{-5}T^2 - 8.2467 \times 10^{-7}T^3 \\
+ 5.3875 \times 10^{-9}T^4)S + (-5.72466 \times 10^{-3} \\
+ 1.0227 \times 10^{-4}T - 1.6546 \times 10^{-6}T^2)S^{3/2} \\
+ 4.8314 \times 10^{-4}S^2) \quad (8.3)
\]

where \( \rho_w \) is the density of pure water given by

\[
\rho_w(T,0) = 999.842594 + 6.793952 \times 10^{-2}T - 9.095290 \times 10^{-3}T^2 \\
+ 1.001685 \times 10^{-4}T^3 - 1.120083 \times 10^{-6}T^4 \\
+ 6.536332 \times 10^{-9}T^5. \quad (8.4)
\]

These equations are empirical best fits to a large quantity of data and should not be viewed as theoretical results. Also, keep in mind that \( \rho_{sw} \) is specifically for standard mean ocean water and not intended for use on other salt solutions, such as NaCl; (8.3) gives up to a 5% error for the density of salt solutions of NaCl, measured using a conductivity meter calibrated in psu.
Example 8.1:
Density gradients in idealized reservoirs.

To illustrate the nonlinear dependence of density on temperature and salinity, imagine a hypothetical reservoir with linear temperature and salinity profiles. The temperature varies from 10 °C at the reservoir bottom to 20 °C at the top, and the salinity varies from 35 psu at the reservoir bottom to 25 psu at the top. Calculate the resulting density variation and density gradient as a function of non-dimensional depth.

The density varies according to (8.3). From a program in Matlab, we calculate the density for one hundred intermediate points between $z/h = 0$ and 1, where $h$ is the reservoir depth. We also calculate the density gradient by a central difference finite difference approximation. The resulting profiles are depicted in Figure 8.2. As we will see in later, the buoyancy frequency, given by $N = \sqrt{(-g/\rho_r)(\partial \rho/\partial z)}$ is a typical measure of the density stratification. Taking $g$ and $\rho_r$ as constants, $N$ will vary in the same way as the density gradient depicted in the figure. Although
the density profile looks linear, the variation of the density gradient highlights the nonlinearity of the density profile. Because linear concentration gradients do not imply linear density gradients, it is important to always work with the appropriate equation of state in stratified fluids.

8.2 Hydrostatics

In the presence of a gravitational field $\mathbf{g}$, density differences result in variable buoyant forces throughout a fluid and a strong feedback to the dynamical equations. An example of this feedback that we know from experience is that hot air rises. This really means that light fluid will tend to move above heavy fluid. In other words, in the presence of gravity, a density stratified fluid has a preferred organization.

Yih (1980) points out that without gravity, the heterogeneity of a fluid can have only minor effects on its behavior. As an anecdote to this case, during the Apollo 13 mission to the moon, the main systems had to be turned off to conserve energy and things got very cold in the capsule. The astronauts were able to keep warm if they stayed very still since they would heat up a cushion of air surrounding their bodies that then acted as insulation against the cold. Without gravity, the warm air does not rise, but rather stays right where it is. But since we are not often concerned with space flight, we must consider the effects of gravitation.
Back in the presence of gravity, the preference to have heavy fluid below light fluid allows us to describe the characteristic stability states of hydrostatic stratified fluids. Shown in Figure 8.3 the density structure can lead to three possible stability states. Stably-stratified fluids tend to maintain their structure, and the stabilizing forces that enforce this preferred organization are the main topics of the dynamics of stratified fluids discussed in the remaining chapters. For neutrally stable fluids there is no preferred structure and fluid particles experience similar resistances to motion in all directions. Unstable stratification occurs when heavy fluid overlays light fluid. Although such a density structure is theoretically possible, it is very sensitive to small scale disturbances and is said to be unstable—it will tend toward the stable state. From a stability analysis that includes surface tension, disturbances of less than 9 mm for water overlaying air are stable. Such small disturbances are hard to prevent; thus, for our purposes we will call stratification distributions where $\frac{\partial \rho}{\partial z} > 0$ unstable.

To compute the pressure distribution in a static, stably-stratified fluid, we start with the hydrostatic pressure equation as is well know from fluid mechanics, giving us

$$\frac{\partial p_s}{\partial z} = -\rho g$$

where $p_s$ is the hydrostatic pressure. For an incompressible fluid $\rho$ is not a function of pressure, and we can rearrange (8.5) to give

$$p_s(z, t) = p_0(0, t) - g \int_0^z \rho(z', t)\,dz'$$

(8.6)
where $p_0$ is the background pressure at the origin of the $z$-axis (recall $z$ is defined vertically upward). Fluids in motion have an excess or deviation pressure in addition to the hydrostatic pressure which drives the motion. In this case,

$$p(z, t) = p_s(z, t) + p'(z, t)$$

where $p'$ is called the dynamic pressure. Technically, it is only appropriate to define a dynamic pressure when the density is constant $\rho(z, t) = \rho_0$. However, when density deviations are small, we often will utilize the dynamic pressure by applying the Boussinesq approximation, described in detail in .

To calculate the pressure distribution for compressible fluids, the density is a function of pressure and an appropriate equation of state must be used. A common model for the behavior of a compressible fluid is the Ideal Gas Law.

### 8.3 Dynamic Equations

For the solution to hydrodynamic problems, it is often very convenient to write the dynamic equations for properties that change “as we follow a fluid particle.” This is done using the material derivative, and we pause here to show where it comes from. Consider a quantity, $\chi$, that is variable throughout the flow field

$$\chi = f(x, y, z, t).$$

We can parameterize $\chi$ so that we follow streamlines through the fluid. In that case, the streamlines have coordinates that are purely functions of time, and the parameterized function for $\chi$ is

$$\chi = f(x(t), y(t), z(t), t).$$

Such a frame of reference, where we are moving with a fluid particle, is called the Lagrangian frame of reference. Generally, we prefer the Eulerian frame of reference which allows us to stay at a single point. To write the total derivative of $\chi$ in the Eulerian reference frame, we use the Chain Rule

$$\frac{d\chi}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

$$= \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i}$$

where we have used the Einsteinian notation of repeated indices to indicate the vector multiplication in the final line. We call the result in (8.10) the material derivative and use it often enough that we define a new material derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}.$$
The material derivative operator (also called the substantial derivative or total derivative) can act on any continuous quantity, such as density, temperature, vector velocity, vector vorticity, etc. The material derivative gives the Eulerian representation of the variation of any quantity “as we follow the fluid.”

8.3.1 Mass conservation

The general equation of mass conservation for a fluid of variable density is known from fluid mechanics (e.g. Currie 2003, Kundu & Cohen 2004) and is given by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0.
\]

(8.12)

The first term is just the local change in density over time due to external processes. The second term represents the local change in density due to the flux of fluid of different density from neighboring points and is called the gradient transport.

If we expand the gradient transport term, we can write mass conservation with the material derivative operator, giving

\[
\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0.
\]

(8.13)

The first term is the change in density moving with the fluid (e.g. due to pressure changes), and the second term contains the divergence of the velocity.

For an incompressible fluid we have the incompressibility condition

\[
\frac{D\rho}{Dt} = 0
\]

(8.14)

because we are following a fluid particle who’s composition does not change and which cannot be compressed. This leads to the continuity equation

\[
\frac{\partial u_i}{\partial x_i} = 0.
\]

(8.15)

Thus, for an incompressible fluid, mass conservation is given simply by the continuity equation (8.15).

8.3.2 Momentum conservation

Momentum conservation for an isotropic Newtonian fluid is given by the well known Navier-Stokes equations. For an incompressible fluid, the Navier-Stokes equations are (Currie 2003)

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + f_i + \nu \frac{\partial^2 u_i}{\partial x_j^2}
\]

(8.16)
where \( \nu \) is the fluid kinematic viscosity. This equation is second-order in space, which gives us enough degrees of freedom to enforce no-slip boundary conditions on the velocity at solid boundaries. Very often, however, we make the inviscid approximation and neglect the friction forces. This results in the Euler equation

\[
\frac{D u_i}{D t} = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + f_i. \tag{8.17}
\]

The Euler equation is only first-order in space; thus, we can no longer enforce the no-slip boundary condition and still have fluid motion.

We can use (8.17) to show that the dynamic pressure drives the fluid motion. If we write the pressure and density variation as the sum of a hydrostatic term and a dynamic term

\[
p(x_i, t) = p_s(x_i) + p'(x_i, t) \tag{8.18}
\]

\[
\rho(x_i, t) = \rho_s(x_i) + \rho'(x_i, t) \tag{8.19}
\]

and we let the body force be due to gravity \( f_i = -\rho g_i = (0, 0, -\rho g) \), we have from the hydrostatic condition (8.5)

\[
\frac{\partial p_s}{\partial x_i} = -\rho_s g_i \tag{8.20}
\]

and substituting into (8.17) we obtain

\[
\rho \frac{D u_i}{D t} = - \frac{\partial (p_s + p')}{\partial x_i} - (\rho_s + \rho')g_i
\]

\[
= - \frac{\partial p_s}{\partial x_i} - \rho_s g_i - \frac{\partial p'}{\partial x_i} - \rho' g_i
\]

\[
= - \frac{\partial p'}{\partial x_i} - \rho' g_i. \tag{8.21}
\]

So we see that motions of stratified fluids are the result of density and pressure deviations from hydrostatic equilibrium.

### 8.3.3 Vorticity conservation

Vorticity is an important quantity in fluid dynamics, and is defined as

\[
\zeta = \nabla \times u = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}
\]

\[
= -\epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{8.22}
\]

where the epsilon operator, called the alternating tensor or the permutation symbol, is defined as

\[
\epsilon_{ijk} = \begin{cases} 
0 & \text{if } i = j, j = k, \text{ or } i = k, \\
1 & \text{if } i < j < k, \\
-1 & \text{otherwise}
\end{cases} \tag{8.23}
\]
The equation of vorticity conservation for an incompressible, Newtonian fluid is given by

\[
\frac{\partial \zeta_i}{\partial t} + u_j \frac{\partial \zeta_i}{\partial x_j} = \varepsilon_{ijk} \frac{\partial p}{\rho^2} \frac{\partial \rho}{\partial x_k} + \zeta_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \zeta_i}{\partial x_j^2}.
\] (8.24)

For irrotational flow:

\[
\frac{D \zeta_i}{Dt} = 0.
\] (8.25)

In a stratified fluid \( \partial \rho/\partial x_k \) is non-zero so that the production term is never zero and we always have vorticity. Therefore, stratified fluids are never irrotational. As we will see in several sections, however, if we are careful about our definitions, we can still make the irrotational flow assumption and make use of the velocity potential.

### 8.3.4 Approximations

Two general approximations will be drawn upon throughout this text. The first of these is the Boussinesq approximation which states for a gradually varying density field that only density differences must be retained; individual occurrences of the density can be replaced with an average, reference density. Returning to the modified Euler equation (8.21), we set the density multiplying the acceleration terms (left hand side) equal to a constant reference density and divide, yielding

\[
\frac{Du_i}{Dt} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x_i} - \frac{\rho' g_i}{\rho_0}.
\] (8.26)

We recall from (8.19) that \( \rho' = \rho - \rho_0 \). Then we define the reduced gravity \( g' \) as

\[
g' = \frac{\rho_0 - \rho}{\rho_0} g.
\] (8.27)

Thus, Euler’s equation becomes

\[
\frac{Du_i}{Dt} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x_i} + g'_i.
\] (8.28)

From this equation we see that for Boussinesq flows, the effect of density differences is to reduce the effective gravitational acceleration. Hence, the motions of stratified fluids appear to take place in a reduced gravitational field so that we will expect the possibility of large amplitude oscillations moving in slow-motion. This is particularly the case for internal waves.

The second typical approximation we will make is that the disturbances from a reference equilibrium state are small. This approximation allows us to linearize the above equations and, thereby, neglect the convective acceleration terms (see the following section on scales).
8. Concepts, Definitions, and Governing Equations

Isopycnals

\[ \rho(z) = \rho_0 (1 - \varepsilon' z) \]

\( \varepsilon' : \) Density Gradient

\[ = -\frac{1}{\rho_0} \frac{d\rho}{dz} \]

Fig. 8.4. Definitions sketch for estimating the natural frequency of a linear stratification profile.

8.4 Parameters

Stratified fluid motions are often described in terms of a few dominant parameters and non-dimensional numbers. The most common of these is the buoyancy frequency. Consider the linear stratification depicted in Figure 8.4. What happens if we displace a small volume of fluid, \( \delta V \), by a small amount, \( \delta z \)? When the fluid reaches the new location, say above the starting point, it is heavier than the surrounding fluid and experiences a restoring (buoyant) force directed back to the starting position. If we use Newton’s second law to estimate the acceleration of the particle, we have

\[ F_i = ma_i \]

\[ -(\rho_0 \varepsilon' z)g(\delta V) = (\rho_0 \delta V) \frac{d^2 z}{dt^2} \]

\[ \frac{d^2 z}{dt^2} + \varepsilon' gz = 0 \quad (8.29) \]

where \( \varepsilon = -1/\rho_0 (d\rho/dz) \). This equation has the same form as the equation that describes one-dimensional simple harmonic motion. The coefficient \( \varepsilon' g \) gives the square of the eigenfrequency of the motion. We will prefer to work with the eigenfrequency itself, and we give it the name buoyancy frequency, or Brunt-Vaisälä buoyancy frequency, defined as

\[ N = \left( -\frac{g}{\rho_0} \frac{d\rho}{dz} \right)^{1/2} \quad (8.30) \]

named after a meteorologist and oceanographer. Thus, our packet of fluid moves back to its original location, but because of its acceleration under gravity, it overshoots \( z = 0 \) and starts to decelerate as it enters heavier fluid. Eventually, it swings back and, in the absence of viscous damping, the harmonic motion would continue with frequency \( N \).
Example 8.2:
Buoyancy frequency in idealized reservoirs.

Consider the idealized reservoir in Example 1. The temperature varies from 10 °C at the reservoir bottom to 20 °C at the top, and the salinity varies from 35 psu at the reservoir bottom to 25 psu at the top. Assuming a reservoir depth of 10 m, calculate the resulting variation of the buoyancy frequency in the flow.

From the density and density gradient profiles in Figure 8.2, we can calculate the buoyancy frequency from

\[ N = \sqrt{-\frac{g}{\rho_0} \frac{\partial \rho}{\partial z}} \]  

(8.31)

Taking the reference density at the density at \( z = 5 \) m, Figure 8.5 plots the variation of \( N \) and \( 1/N \). From the figure, we see that the buoyancy frequency increases with depth because the absolute value of the density gradient also increases with depth. Also, the inverse of the buoyancy frequency gives the period of natural oscillation of the stratification, which for this reservoir is of the order 10 s. This type of stratification may be typical of a strongly stratified estuary, but
in most lakes, the salinity variation is small so that the buoyancy frequency is smaller and the
period of oscillation is larger.

Several other important parameters can be derived by non-dimensionalizing the governing
equations of motion. To do this, we define the characteristic scales in the problem to have a time
scale $T$, length scale $L$, velocity scale $U$ and pressure scale $P$ and assume that the body forces
are due to gravity $g$. Then we can define the non-dimensional variables

$$ u = U u^*; \quad t = T t^*; \quad x = L x^*; \quad p' = P p'^*; \quad f = -\rho g g^* $$

We begin by substituting these scales into the continuity equation (8.15)

$$ \frac{1}{L} \nabla^* \cdot (U u^*) = 0 \quad (8.33) $$

Since the characteristic scales are constants, this leads to

$$ \nabla^* \cdot u^* = 0 \quad (8.34) $$

Thus, no parameters are deduced from the conservation of mass. This also implies that we cannot
approximate the conservation of mass, since there are no parameters that we can assume are
small in a given problem.

Next, we turn to the Navier-Stokes equations with the hydrostatic pressure removed. Substituting
the characteristic scales, we have

$$ \frac{U \rho \partial u^*}{T} + \frac{U^2 \rho}{L} u^* \cdot \nabla^* u^* = -\frac{P}{L} \nabla^* p'^* + \frac{U \mu}{L^2} \nabla^* u^* - \rho g g^* $$

To continue, we make the Boussinesq approximation and let $\rho = \rho_0$ and $\rho' = \rho - \rho_0$. We then
multiply the equation by $L/(\rho_0 U^2)$ to obtain a dimensionless equation. This results in

$$ \frac{L}{U T} \frac{\partial u^*}{\partial t^*} + u^* \cdot \nabla^* u^* = -\frac{P}{\rho_0 U^2} \nabla^* p'^* + \frac{\nu}{U L} \nabla^* u^* + \frac{g' L}{U^2} g^* $$

The dominant scales in stratified flows are given by the leading coefficients in the non-dimensional
equation.

The Strouhal number is defined as

$$ St = \frac{L}{U T} = \frac{\text{Local unsteady acceleration}}{\text{convective inertia}} $$

It gives the relative importance of the non-linear convective terms to the overall accelerations in
the flow. $St$ is commonly used in bluff-body flow to estimate the shedding frequency of eddies
in the wake.

The Reynolds number is given by

$$ Re = \frac{UL}{\nu} = \frac{\text{convective inertia}}{\text{viscous stress}} $$

$Re$ compares the importance of friction to the non-linear convective terms. At high $Re$ we
know that the flow becomes turbulent. Hence, it is the non-linearity of the dynamics that are
responsible for turbulence. We also see that flows of a certain scale $L$ such that $Re$ is small are
influenced by friction. Thus, friction damps out small-scale motion.
The most important parameter in this equation to stratified flows is the Richardson number

$$R_i = \frac{g' \, L}{U^2} = \frac{\text{density forces}}{\text{convective inertia}}$$

(8.39)

For a stably stratified flow, $R_i$ summarizes the relative magnitude of the restoring force of the density stratification to the destabilizing effect of the non-linear convective terms. An alternate definition of $R_i$ is given by

$$R_i = \frac{N^2}{(du/dz)^2}$$

(8.40)

and represents the ratio of work required for mixing to the kinetic energy available from a shear velocity profile. This definition is useful in parallel shear flows to assess the potential for mixing, as in a stratified reservoir.

The $R_i$ term can also be recognized as a modified Froude number, which we call the densimetric Froude number

$$F_{r_d} = \frac{U}{\sqrt{g' \, L}} = \frac{1}{\sqrt{R_i}}$$

(8.41)

Since $Fr$ gives the velocity of a gravity wave, we see that internal gravity waves in density-stratified fluids move at a reduced speed $\sqrt{g' \, L}$ due to the reduced gravity term.

Summarizing these parameters, the non-dimensional momentum equation becomes

$$St \, \frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\frac{P}{\rho_0 U^2} \nabla^* p^* + \frac{1}{Re} \nabla^2 \mathbf{u}^* + R_i g^*$$

(8.42)

Summary

This chapter introduced the effects of stratification. Stratification is the layering of a fluid system due to density variations. Stably stratified fluids have increasing density with depth. Since this is the first chapter to discuss dynamics, the governing dynamic equations of fluid motion were introduced. These included equations for mass, momentum, and vorticity conservation. From the equation for the motion of a fluid parcel in a linearly stratified ambient, the buoyancy frequency was defined. Other important parameters describing stratification are the Richardson number and the density-adjusted Froude number, which were derived by non-dimensionalizing the Navier-Stokes equations.

Exercises

8.1 Equations of state. Write a computer program (easiest using Matlab) to calculate the density of standard mean ocean water for any given temperature and salinity.

8.2 Stratified fluid parameters. Show that the densimetric Froude number is proportional to a bulk Richardson number according to the relationship

$$F_{r_d} = \frac{1}{\sqrt{R_i}}$$

Hint, approximate the derivatives using $\Delta \rho$, $u$, and $L$. 


8.3 Natural stratified fluid scales. Estimate values for velocity and density gradients for a pond (depth 1 m), lake (depth 40 m), estuary (mixture of fresh and salt water), and the ocean and use them to calculate typical values of $N$, $1/N$, $\sqrt{gh}$, $Ri$, and $Fr_d$. What role does stratification play in the dynamics of each of these systems?