1 Shallow flow stability equation

Show that the stability of a shallow shear flow is governed by the non-dimensional stability equation

\[(c - u)\phi_v'' + (u'' + k^2(u - c))\phi_v = \frac{S_f u}{ik} \left( \phi_v'' + \frac{u'\phi_v'}{u} - \frac{k^2 \phi_v}{2} \right) + \frac{1}{ikRe} \left( 2k^2 \phi_v'' - \phi_v''' - k^4 \phi_v \right)\]

where the stability number \(S_f\) and Reynolds number \(Re\) are defined as

\[S_f = \frac{c_f b}{h}\]  \hspace{1cm} (2)
\[Re = \frac{ub}{\nu}.\]  \hspace{1cm} (3)

The wave number of the disturbance is \(k\) and the wave frequency is \(\omega; c = \omega/k\). The left-hand-side of this equation is known as the Orr-Sommerfeld equation.

To do this, perform the following calculations.

1. Begin with the governing equations in shallow flows, called the viscous Saint-Vénant equations

\[\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y} = 0\]  \hspace{1cm} (4)

\[\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{c_f}{2h} \sqrt{\tilde{u}^2 + \tilde{v}^2} + \nu \left( \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} \right)\]

\[\frac{\partial \tilde{v}}{\partial t} + \tilde{u} \frac{\partial \tilde{v}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{c_f}{2h} \sqrt{\tilde{u}^2 + \tilde{v}^2} + \nu \left( \frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} \right)\]
where $\tilde{u}$ and $\tilde{v}$ are the depth- and short-time average velocities in the $x$- and $y$-directions and $\tilde{p}$ is the depth- and short-time average dynamic pressure; $\nu$ is the molecular viscosity. Here, we have made the rigid lid assumption to simplify the problem.

Define the perturbation velocities

$$\tilde{u} = U(y) + u(x,y)$$
$$\tilde{v} = v(x,y)$$

where $U(y)$ is the base velocity profile. Substitute this velocity vector into the equations of motion and linearize the equations by neglecting products of the fluctuations.

2. Following classical linear stability analysis, substitute sinusoidal velocity and pressure perturbations of the form

$$(u, v, p) = (\phi_u(y), \phi_v(y), \phi_p(y)) \exp(i(kx - \omega t))$$

where $k$ and $\omega$ are the wave number and wave frequency of the disturbances, and it is assumed that we retain only the real part of the exponential.

3. Algebraically manipulate the equations you obtain to remove $\phi_u$ and $\phi_p$ (Hint: refer to the derivation of Rayleigh’s Instability Theorem for an example of how to do this).

4. Non-dimensionalize the resulting equation by introducing the following dimensionless variables (denoted by superscript stars)

$$U = \pi u^*,$$  
$$c = \pi c^*,$$  
$$\phi = \pi \phi^*,$$  
$$y = b y^*,$$  
$$k = k^*/b$$

where $\pi$ and $b$ are characteristic velocity and length scales of the base velocity profile $U(y)$.

5. If the result you obtain is different than that desired, note the differences and suggest where in the derivation an error or missing assumption might be applied to correct the problem.

2 Inflection Points

Rayleigh’s instability theorem makes a statement about inflection points in shear velocity profiles. The classical shear a jets or wake is

$$U(y) = \pi \left(1 - R + 2R \text{sech}^2 \left(\frac{y}{b}\right)\right)$$

where $y$ is the transverse coordinate, $b$ is a characteristic length scale of the wake, $\pi = \frac{1}{2}(u_c + u_\infty)$, and $R = (u_c - u_\infty)/(u_c + u_\infty)$; $u_c$ is the centerline velocity and $u_\infty$ is the ambient flow velocity. Jets are given by positive values of $R$ and wakes by negative values. The classical shear profile for the mixing layer is

$$U(y) = \pi' \left(1.0 + R' \tanh \left(\frac{y}{b}\right)\right)$$
where

\[ \bar{u}' = \frac{1}{2}(U(\infty) + U(-\infty)) \]
\[ R' = (U(\infty) - U(-\infty))/(U(\infty) + U(-\infty)). \]

Perform the following steps:

1. Plot each profile for \( b = 1 \) over the range \( y = [-3, 3] \).

2. Show that each of these classical profiles has an inflection point (Hint: you can do this numerically by calculating the first and second derivatives numerically (using fine-scale data) or by calculating the derivatives analytically).

3. Compute the velocity gradient \( du/dy \) at the inflection point (again, numerical or analytical results are acceptable).

4. From Rayleigh’s instability theorem, what implication is their of the existence of inflection points in these profiles for the stability of these shear flows?

3 Kelvin-Helmholtz Instability

Repeat the heuristic derivation of the stability criteria for the onset of Kelvin-Helmholtz mixing but without using the Boussinesq approximation. Show that the new stability criterion is

\[ \frac{1}{4} - \frac{1/\rho(d\rho/dz)}{1/U(dU/dz)} + \frac{1}{4} \frac{d\rho}{\rho} > Ri \quad (11) \]

where \( \rho \) and \( U \) are the density and velocity of the lower particle. Hint: the buoyant work is unchanged. The total kinetic energy before exchange is:

\[ \frac{1}{2} \left[ \rho U^2 + (\rho + d\rho)(U + dU)^2 \right] \quad (12) \]

After the exchange, each particle has the mean velocity

\[ U + \frac{dU}{2} \quad (13) \]

but their densities do not mix, so that the total kinetic energy after exchange is

\[ \frac{1}{2} (\rho + \rho + d\rho)(U + dU/2)^2 \quad (14) \]

Follow this notation to complete the derivation.

Compare this result to our original result and discuss the differences from a physics point of view.

4 Lake Mixing

When the wind blows, shear velocity can be set up in the thermocline of a lake which may develop Kelvin-Helmholtz instabilities which result in mixing and deepening of the thermocline. Consider a lake with epilimnion temperature 20°C and hypolimnion temperature 7°C. The depth of the epilimnion is 5 m, the depth of the hypolimnion is 30 m, and the fetch (mean distance over which the wind blows) is 4 km.
1. What value of the velocity gradient at the thermocline will be necessary to initiate mixing by Kelvin-Helmholtz instability?

2. (Extra Credit) What wind-speed $U_{10}$ measured at 10 m above the lake would you expect to generate the velocity gradient computed in Part 1?

3. (Extra Credit) Do you think the wind speed computed in Part 2 is reasonable to occur on a summer's day? Why or why not?