Summary of Cross-Section Properties for Parabolic Splayed

\[ y = kx^2 \]

\[ Y_c = \frac{h}{6} \]

\[ T = \frac{kh^2}{3} \]

\[ I_x = \frac{bh^3}{12} \]

\[ I_y = \frac{bh^3}{6} \]

\[ I_{xc} = \frac{37}{2100} bh^3 \]

\[ I_{yc} = \frac{bh^3}{80} \]

\[ A = \frac{bh}{2} \]

\[ x = \frac{b}{4} \]

\[ y = \frac{3h}{10} \]
Determine the location of the centroid for the shaded area shown.

\[ y = kx^2 \] or \[ x = \sqrt{\frac{y}{k}} \]

\[ A(b, h) \]

First determine the value of \( k \) given coordinates of \( A \).

1. \[ y = kx^2 \]
2. \[ h = kb^2 \]
3. \[ k = \frac{h}{b^2} \]
4. So \[ y = \frac{h}{b^2} x^2 \] or \[ x = b \sqrt{\frac{y}{h}} \]
Calculate the area.

(5) \[ dA = y \, dx \]

(6) \[ dA = \frac{h}{b^2} \, x^2 \, dx \]

(7) \[ A = \frac{h}{b^2} \int_0^b x^2 \, dx \]

(8) \[ A = \frac{h}{b^2} \left( \frac{b^3}{3} \right) \]

(9) \[ A = \frac{bh}{3} \]

Distance from x-axis to centroid of diff. element.

Calculate \( \int y \, dA \)

(10) \[ \int y \, dA = \int_0^b \frac{1}{2} y \frac{h}{b^2} \, x^2 \, dx \]

(11) \[ = \int_0^b \frac{1}{2} \left( \frac{h}{b^2} \right) \, x^2 \, dx \]

(12) \[ = \frac{b^4}{2} \int_0^b x^4 \, dx \]

(13) \[ = \frac{h^2}{2} b^4 \left( \frac{1}{5} \right) b^5 \]
<table>
<thead>
<tr>
<th>(14)</th>
<th>$\int y , dA = \frac{h^2 b}{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calculate $\overline{y}$</td>
</tr>
<tr>
<td>(15)</td>
<td>$\overline{y} = \frac{\int y , dA}{\int dA}$</td>
</tr>
<tr>
<td></td>
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<tr>
<td>(16)</td>
<td>$\overline{y} = \frac{h^2 b}{10 , b , h}$</td>
</tr>
<tr>
<td>(17)</td>
<td>$\overline{y} = \frac{3 , h^2 b}{10 , b , h}$</td>
</tr>
<tr>
<td>(18)</td>
<td>$\overline{y} = \frac{3}{10} , h$</td>
</tr>
<tr>
<td></td>
<td>Calculate $\int x , dA$</td>
</tr>
<tr>
<td>(19)</td>
<td>$\int x , dA = \int_0^b x , \frac{h}{b^2} , x^2 , dx$</td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>(20)</td>
<td>$= \frac{h}{b^2} \int_0^b x^3 , dx$</td>
</tr>
<tr>
<td>(21)</td>
<td>$= \frac{1}{4} , \frac{h}{b^2} \int_0^b x^4 , dx$</td>
</tr>
</tbody>
</table>

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\[
\begin{align*}
(22) \quad \sum x dA &= \frac{h b^4}{4 b^2} \\
(23) \quad \sum x dA &= \frac{h b^2}{4} \\
(24) \quad \bar{x} &= \frac{\sum x dA}{\sum dA} \\
(25) \quad \bar{x} &= \frac{h b^2}{4} \\
(26) \quad \bar{x} &= \frac{3 h b^2}{4 b h} \\
(27) \quad \bar{x} &= \frac{3}{4} b \\
\end{align*}
\]

\[
\bar{x} = \frac{3}{4} b \quad \text{and} \quad \bar{y} = \frac{3}{10} h
\]
Calculate the moments of inertia for the shaded area shown about the X and Y axes.

\[ y = kx^2 \]

\[ y = \frac{h}{b^2} x^2 \quad \text{or} \quad x = b \sqrt{\frac{y}{h}} \]

Calculate \( I_y \)

choose a differential element that is parallel to the Y axis as shown.
\( y = \frac{h}{b^2} x^2 \)

\( dA = y \, dx \)

\( dA = \frac{h}{b^2} x^2 \, dx \)

\begin{align*}
(2) \quad I_y & = \int x^2 \, dA \\
(3) \quad I_y & = \int_0^b x^2 \left( \frac{h}{b^2} x^2 \right) \, dx \\
(4) \quad I_y & = \frac{h}{b^2} \int_0^b x^4 \, dx \\
(5) \quad & = \frac{h}{b^2} \left[ \frac{1}{5} x^5 \right]_0^b \\
(6) \quad I_y & = \frac{h}{b^2} \left( \frac{1}{5} b^5 \right) 
\end{align*}
\( I_y = \frac{h b^3}{3} \)

**Calculate** \( I_x \)

Choose a differential element that is parallel to the \( x \)-axis as shown.

\[
X = b \sqrt{\frac{y}{h}}
\]

\[ (b-x)dy \]

\[ dy \]

\[ dy \]

\[ dy \]

\[ y \]

\[ y \]

\[ y \]

\[
I_x = \int y^2 dA
\]

\[
I_x = \int_0^h y^2 (b-x) dy
\]

\[
I_x = \int_0^h y^2 (b-b \sqrt{\frac{y}{h}}) dy
\]
\[ I_x = \int_0^h y^2 \, dy - \int_0^h \frac{y}{h} \sqrt{y} \, dy \]

\[ I_x = b \int_0^h y^2 \, dy - \frac{b}{h} \int_0^h y^{3.5} \, dy \]

\[ = b \left[ \frac{1}{3} y^3 \right]_0^h - \frac{b}{h} \left[ \frac{1}{3.5} y^{3.5} \right]_0^h \]

\[ = \frac{bh^3}{3} - \frac{bh^{3.5}}{3.5} \]

\[ = \frac{76bh^3}{21} - \frac{6bh^3}{21} \]

\[ I_x = \frac{bh^3}{21} \]

Calculate \( I_x \) with alternate method. This method will use a vertical diff. element instead of the horizontal diff element as shown.
Because the differential element is not parallel to the x-axis, then we have to first express the moment of inertia of the differential element about the x-axis.

It is observed that the base of the differential element touches the x-axis for the entire area.

The moment of inertia of a rectangular area about its own base is given as follows.
Therefore the moment of differential moment of inertia associated with the differential element is

\[ y = \frac{h}{b^2} x^2 \]

\[ dI_x = \frac{1}{3} \left( \frac{h}{b^2} x^2 \right)^3 dx \]  \hspace{1cm} (19)

\[ dI_x = \frac{1}{3} \frac{h^3}{b^6} x^6 dX \]  \hspace{1cm} (20)

\[ = \int y^3 \, dx \]

Exploded View of Differential Element

The total moment of inertia is then given by integrating the differential moment of inertia across the area.

\[ I_x = \int_0^b dI_x \]  \hspace{1cm} (21)
\( (22) \quad I_x = \int_0^b \left( \frac{1}{3} \cdot \frac{h^3}{b^6} \right) x^6 \, dx \)

\( (23) \quad I_x = \frac{h^3}{3 b^6} \int_0^b x^6 \, dx \)

\( (24) \quad I_x = \frac{h^3}{3 b^6} \frac{1}{7} x^7 \bigg|_0^b \)

\( (25) \quad I_x = \frac{h^3 b^7}{3 b^6} \)

\( (26) \quad I_x = \frac{b h^3}{21} \) (Same answer)

Calculate \( I_x \) with another alternative method.

This method will use a vertical differential element as with the first alternative solution. However, advantage will not be taken of the fact that bottom of the differential element lines up with the \( x \)-axis. In this case the parallel axis theorem will have to be used to set up the expression for \( I_x \) as follows,
The moment of inertia of the differential element about its own centroid is given as follows.
<table>
<thead>
<tr>
<th>(27)</th>
<th>$d\bar{I}_x = \frac{1}{12} (d\bar{I}_x)^{\frac{3}{2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(28)</td>
<td>$d\bar{I}_x = \frac{1}{12} \left( \frac{h^x}{b^2} \right)^{\frac{3}{2}} d\bar{I}_x$</td>
</tr>
<tr>
<td>(29)</td>
<td>$d\bar{I}_x = \frac{h^x}{12 b^6} d\bar{I}_x$</td>
</tr>
</tbody>
</table>

Then, the moment of inertia of the differential element about the $x$-axis is found using the parallel axis theorem as follows:

| (30) | $d\bar{I}_x = d\bar{I}_x + dA \bar{z}^2$ |
| (31) | $d\bar{I}_x = \frac{h^3 x^6}{12 b^6} d\bar{I}_x + \left( y d\bar{I}_x \right) \left( \frac{y}{2} \right)^2$ |
| (32) | $d\bar{I}_x = \frac{h^3 x^6}{12 b^6} d\bar{I}_x + \left( \frac{h}{b^2} X d\bar{I}_x \right) \left( \frac{h}{2b} X \right)^2$ |
| (33) | $d\bar{I}_x = \frac{h^3 x^6}{12 b^6} d\bar{I}_x + \frac{3 x^6}{4 b^6} d\bar{I}_x$ |
| (34) | $d\bar{I}_x = \frac{h^3 x^6}{b^6} \left[ \frac{1}{12} + \frac{1}{4} \right] d\bar{I}_x$ |
| (35) | $d\bar{I}_x = \frac{h^3 x^6}{3 b^6}$ |
The differential moment of inertia expressed in equation (35) is exactly the same as that expressed in equation (20). Thus the desired moment of inertia is as given in equation (26).

\[ I_x = \frac{bh^3}{2} \]

\[ I_x = \frac{bh^3}{12} \quad \text{and} \quad I_y = \frac{h b^3}{3} \]
Calculate the moments of inertia for the shaded area shown about centroidal axes that are parallel to the x and y axes.

\[ y = kx^2 \]

\[ \overline{x} = \frac{3}{4} b \]
\[ \overline{y} = \frac{3}{8} h \]

Given from previous example:

\[ A = \frac{bh}{3} \]
\[ I_x = \frac{bh^3}{24} \]
\[ I_y = \frac{bh^3}{2} \]

Calculate \( I_{xc} \) using parallel axis theorem.
<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>[ I_x = I_{xc} + A \frac{d^2}{2} ]</td>
</tr>
<tr>
<td>(2)</td>
<td>[ \frac{bh^3}{2l} = I_{xc} + \frac{bh}{3} \left( \frac{3}{10} \frac{h}{l} \right)^2 ]</td>
</tr>
<tr>
<td>(3)</td>
<td>[ I_{xc} = \frac{bh^3}{2l} - \frac{9}{300} bh^3 ]</td>
</tr>
<tr>
<td>(4)</td>
<td>[ I_{xc} = \frac{bh^3}{2l} \left( \frac{300 - 9}{12l} \right) ]</td>
</tr>
<tr>
<td>(5)</td>
<td>[ I_{xc} = \frac{111 bh^3}{6300} ]</td>
</tr>
<tr>
<td>(6)</td>
<td>[ I_{xc} = \frac{37 bh^3}{2100} ]</td>
</tr>
</tbody>
</table>

### Calculate \( I_y \) using parallel axis theorem

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7)</td>
<td>[ I_y = I_{yc} + A \frac{d^2}{2} ]</td>
</tr>
<tr>
<td>(8)</td>
<td>[ \frac{bh^3}{5} = I_{yc} + \frac{bh}{3} \left( \frac{3}{4} b \right)^2 ]</td>
</tr>
<tr>
<td>(9)</td>
<td>[ I_{yc} = \frac{bh^3}{5} - \frac{9}{48} bh^3 ]</td>
</tr>
<tr>
<td>(10)</td>
<td>[ I_{yc} = \frac{bh^3}{240} \left( 48 - 45 \right) ]</td>
</tr>
<tr>
<td>(11)</td>
<td>[ I_{yc} = \frac{bh^3}{240} ]</td>
</tr>
</tbody>
</table>

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\[
\int y_c = \frac{h b^3}{80}
\]