Pile-Driving Analysis by the Wave Equation

By E. A. L. Smith

With discussion by Messrs. L. O. Soderberg; Marvin Gates; E. Jonas; A. A. G. M. Cornfield; Robert D. Chells and Arthur F. Zaskey; and E. A. L. S

SYNOPSIS

There are a great many different pile-driving formulas in use, and engineers have never been able to agree as to which one is best. This situation has arisen primarily because, until recently, the mathematics of pile-driving action could not be solved in any practical manner. As a result all pile-driving formulas are partly empirical and, consequently, apply only to certain types or lengths of pile. This paper is presented with the purpose of giving engineers a mathematical method of wider application, depending on the use of electronic computers and numerical integration. The method is also applicable to other impact problems.

INTRODUCTION

Pile-driving formulas are widely used to determine the static bearing capacity of piles. Some of these formulas are also used to determine stresses in the pile during driving.

An astonishing amount of effort and ingenuity has been expended by engineers in the development of pile-driving formulas, with the result that there are a great many different formulas in use, many of which are specified in various building codes throughout the United States and abroad. Robert D. Chells lists thirty-eight pile driving formulas, and the editors of Engineering News-Record have on file four hundred and fifty such formulas.

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The fact that engineers have been unable to agree on any one pile-driving formula is understandable because, until recently, the mathematics of pile-driving action could not be solved in any practical manner. Pile driving is not a simple problem of impact that may be solved directly by Newton's laws. Pile driving is a problem in longitudinal wave transmission that is covered in a general way by the wave equation, which is well known to mathematicians.

Furthermore, pile driving involves many complications such as the use of capblocks, pile caps, cushion blocks, composite piles, and tapered piles, as well as the elastic-plastic action of the ground and other problems in soil mechanics. As a result of these difficulties, all pile-driving formulas are partly empirical and consequently apply only to certain types or lengths of pile. The paper is presented with the purpose of giving engineers a mathematical method of wider application.

So far as the writer is aware, D. V. Isaacs, in 1931, was the first to point out that wave action occurred during the driving of piles. In 1938, E. N. Fox published a solution of the wave equation applied to pile driving, but, as no electronic computers were available at that time, he was forced to use a number of simplifying assumptions that lessened the value of his solution. At the present time (1960), by using the conceptions of the wave equation and resorting to numerical integration and electronic computers, a solution of the pile-driving problem can be obtained that produces mathematical accuracy within about 5%. This degree of accuracy is more than sufficient in view of our present imperfect knowledge of the physical conditions involved.

The mathematical method described herein may, with slight modification, be applied to other impact problems such as the design of a foundation for a forging hammer, or a fendering system for a dock.

OUTLINE OF NUMERICAL METHOD

The hammer, pile, and other parts involved, such as the capblock and pile cap, are represented as a series of weights and springs as shown in Fig. 1, and the time during which the action occurs is divided into small time intervals such as 1/4,000 sec. The action of each weight and each spring is then calculated separately in each and every time interval. In this way a mathematical determination may be made of stresses, and of pile penetration or permanent set per blow, against any amount or kind of ground resistance.

The process may be compared to making drawings for an animated motion picture. The artists who prepare such drawings must take account of the fact that the film will be projected at 24 frames per sec. Each drawing must therefore differ from the preceding drawings by 1/24 sec. In order that the picture may appear realistic when projected, computations must be made to determine how far any moving object will progress in each 1/24 sec. If the motion is uniform, the displacements in each succeeding drawing should be uniform; if the motion is uniformly accelerated, as in the case of a falling object, the displacements should differ by increasing amounts readily calculated by using the well-known laws of uniformly accelerated motion.

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Obviously the rules for making such a set of motion-picture drawings are these:

1. Time is divided into intervals of 1/24 sec each.
2. No motion can be shown in any one drawing.
3. Motion is depicted by making each successive drawing differ from the preceding drawing by just enough to represent the changes occurring during one interval.

The rules for making a numerical pile calculation are almost identical, and may be stated thus:

a. Time is divided into small intervals such as 1/4,000 sec.
b. It is assumed that all velocities, forces, and displacements will have fixed values during any particular interval.
c. The velocities, forces, and displacements for each interval will be computed so as to differ from those existing in the preceding interval by just enough to represent the change occurring during one interval.

If these two sets of rules are compared rule by rule, it will be found that there is no essential difference.

**DIAGRAMMATIC REPRESENTATION**

The hammer-ram and the pile cap are usually short, heavy, rigid objects which may be represented as individual weights without elasticity. In Fig. 1 the first weight \( W_1 \) represents the ram, and the second weight \( W_2 \) represents the pile cap.

The capblock is a short springy object of wood, plastic, or similar material which is comparatively light and which may therefore be represented by a spring. In Fig. 1 this spring is numbered \( K_1 \).

The pile is a heavy object, but is somewhat compressible because of its length. It is therefore subject to wave action under the blow of the hammer. This wave action can be analyzed mathematically by dividing the pile into short sections or "unit lengths" such as 5 ft or 10 ft in length. The weight of each unit length is represented by an individual weight (\( W_3 \) to \( W_{12} \) in Fig. 1) and the elasticity of each unit length is represented by an individual spring (\( K_2 \) to \( K_{11} \) in Fig. 1). The motion of each weight and each spring is then calculated as though each were actually a separate and distinct object.

If the pile is of uniform section, the weights and springs representing it are all alike. If the pile is tapered or is a composite pile, or is in any way irregular in cross section, then these weights and springs differ from one another so as to represent approximately the actual distribution of weight and elasticity throughout the length of the pile.

One may wonder at the fact that comparatively large unit lengths, such as 5 ft or 10 ft, give desired accuracy in computing the action of the impact wave. The following analogy, using water waves, may make the matter clear. Of course impact waves in a pile are longitudinal waves, whereas water waves are transverse waves; nevertheless, water waves can be used to illustrate the principles involved.

Imagine a body of water on which moderately long waves are traveling from right to left. If a long strip of flexible material such as sponge rubber is allowed to float on the surface, it will follow the wave action exactly. If for purposes of mathematical analysis we were to represent this flexible strip by a number of short rigid floats connected together by flexible links, the picture would appear as shown in Fig. 2. This would be an approximation, but the approximation would involve negligible error because the small floats would ride the waves almost exactly like the flexible strip.

But suppose we make the rigid floats comparatively long so that they approach or exceed the length of the water wave, then the picture would look like Fig. 3. These long floats cannot follow the wave form closely. If the floats were made longer still, the resemblance to the true wave form would disappear completely.

From the foregoing, it may be concluded that in dividing a pile into unit lengths for purposes of calculation, the unit lengths chosen must be considerably shorter than the wavelength of the stress or impact wave produced by the hammer. Fortunately, a pile-driving impact usually produces a fairly long wave form, therefore division of the pile into lengths of the order of 5 ft to 10 ft will produce acceptable accuracy.

In the special and unusual case where the exact form taken by the impact wave is being investigated, a smaller unit length such as 2 ft or even 1 ft may be advisable. In this case, it may be advisable to divide the ram into unit lengths as well as the pile. In this way a high degree of accuracy may be obtained.

**FIG. 2.—SHORT FLOATS**

It has been explained previously, that the pile must be divided into unit lengths and that time must be divided into small intervals such as 1/4,000 sec. There is a very important relationship between these two procedures. The smaller the unit lengths chosen, the smaller must be the time interval.

This can be explained in a general way by again considering a motion picture. If the action to be represented is of the normal kind, a 1/24-sec time interval between pictures is suitable. If the action is high speed in character, such as the rotation of a wheel, a much smaller interval is required to show the action accurately. If a 1/24-sec interval is used, the wheel may actually appear to run backward. The same sort of phenomenon occurs in a numerical wave equation calculation because the calculation is a step-by-step process which must keep ahead of the stress wave caused by the hammer blow. The smaller the unit length is made, the smaller must be the time interval. It is also true that especially large or suddenly applied external forces or resistances may require the use of a smaller time interval.

It follows from the foregoing, that whenever a numerical wave equation calculation is made, care must be taken to make certain that the time interval used is not too large. On the other hand, the time interval should not be unnecessarily small, because this would use an unnecessarily large amount of time in computation with little or no increase in accuracy. Consequently, the choice of the correct time interval becomes an important consideration. For
pile calculations, a time interval of about 1/4,000 sec will usually be found to
be satisfactory. This matter is discussed further in the Appendix.

GENERAL

The foregoing discussion outlines the basic method for obtaining a numerical
solution of the pile-driving problem. This method has the outstanding advantage
of dividing the problem into a number of simpler problems each of which may
be considered more or less independently.

Before going into the mathematics, let us first consider the practical or
physical aspects of the problem.

Soil Mechanics.—In order to make a pile calculation, it must be assumed
that the soil will act in some particular way. When future investigators de-
velop new facts, the mathematical method explained herein can be modified
readily to take account of them, but on the basis of information presently avail-
able, the assumptions listed in what follows are recommended.

Resistance At Point Of Pile.—Chellis gives a precedent to follow, which
 teaches that the ground compresses elastically for a certain distance, (which

![FIG. 3.—LONG FLOATS](image)

Chellis calls $C_0$, but which herein will be termed $Q$ for “quake”) and then fails
plastically with constant or “ultimate” resistance $R_u$. This concept is illus-
trated in Fig. 4.

Starting at $O$, the pile point moves ahead a distance $Q$ (usually assumed to
be 0.1 in.) compressing the soil elastically so that at point $A$ the ground resis-
tance $R_x$ has built up to its ultimate value $R_u$. Plastic failure then occurs and
ground resistance remains equal to $R_u$ until the pile point reaches $B$. Elastic
rebound equal to $Q$ then occurs, and motion ceases at point $C$ where all forces
are zero. The permanent set of the pile is the distance $s = OC = AB$.

This conception fails to consider the element of time. Some piles penetrate
the ground more rapidly than others. Obviously, the ground will offer more in-
stantaneous resistance to rapid motion than to slow motion. We therefore in-
troduce the additional factor of “viscous damping,” which is commonly used in
vibration problems.

The numerical wave equation calculation gives the instantaneous velocity of
the point of the pile in any time interval. If this instantaneous velocity is
called $v_p$, and if $J$ is a damping constant, then the product $J v_p$ may be used to
increase (or decrease) the ground resistance so as to produce damping. Thus
at any point $x$ on the line OABC of Fig. 4, the instantaneous damping resistance
is $J v_p R_x$. If $J$ is assumed to have a value of 0.15, and if at point $x$ the cal-
culated value of $v_p$ is 2 fps, the damping resistance is $0.15 \times 2 \times R_x = 0.30 \times R_x$,
and the total instantaneous resistance is $R_x + 0.30 \times R_x = 1.30 \times R_x$.

![FIG. 4.—STRESS-STRAIN DIAGRAM AT PILE POINT](image)

The damping resistance is, of course, temporary or instantaneous, and does
not contribute to the bearing capacity of the pile. It should also be noted that
the constant $J$ refers only to resistance at the point of the pile, such as $R_{12}$ of
Fig. 1.

Thus we see that resistance at the point of the pile is calculated to take ac-
count of the following:

1. Elastic ground compression or quake “$Q$.”
2. Ultimate ground resistance “$R_u$.”
3. Viscous damping based on a damping constant “$J$.”

Resistance Alongside The Pile.—Resistance alongside the pile is calculated
like resistance at the point of the pile, except that a damping constant $J'$ is
used instead of the damping constant $J$ which has already been discussed in
connection with resistance at the point. Thus in Fig. 1 $J'$ would apply to re-

distances $R_3$ to $R_{11}$ inclusive, whereas $J$ would apply only to the point re-
sistance $R_{12}$.

As the pile is driven downward, the soil under the point of the pile is dis-
placed or caused to flow aside very rapidly. However, the soil alongside the
pile is not correspondingly displaced. The value of $J'$ should therefore be
smaller than the value of $J$. A value of $J' = 0.05$ will be assumed. (This com-

pares with a value of 0.15 assumed for $J$).

In the numerical wave equation calculation the ground resistance may be
distributed over the full length of the pile in any way that is desirable. For
simplicity all resistance may be considered to be concentrated at the point of
the pile, but the foundation engineer is at liberty to adopt any distribution that
in his judgment is best suited to the specific ground conditions as disclosed by
borings or other soil investigations.

Physics.—In addition to the action of the soil, we must consider the physical
characteristics of the hammer, capblock, pile cap or follower, and pile. Pre-
cast concrete piles also require a cushion on the head of the pile. The physical
characteristics of this cushion must also be considered. Each of these elements will now be discussed separately.

Hammer.—The hammer ram is ordinarily a short, heavy, rigid object that can be represented by a single weight without elasticity, such as \( W_1 \) of Fig. 1. In special cases where the ram is comparatively long and slender, or where an especially accurate stress analysis is required, it may be advisable to divide the ram into a number of weights and springs (as shown in Fig. 10(b) to be presented subsequently).

The velocity of the ram at the instant of impact is needed in order to start the numerical calculation. Ordinarily the rated foot pounds of energy of the hammer is given by the manufacturer. The efficiency is sometimes given and

\[
\text{Stress} \quad \frac{\text{Strain}}{\text{Stress}} = K
\]

![Stress-Strain Diagram for Capblock](image)

FIG. 5.—STRESS-STRAIN DIAGRAM FOR CAPBLOCK

sometimes must be assumed. From these data the velocity at impact may be calculated by means of the following formula:

\[
\text{Velocity At Impact}, \text{ in fps} = \sqrt{\frac{\text{Rated energy, in ft} - \text{lb} \times \text{Efficiency} \times 64.4}{\text{Weight, in lb}}}
\]

Capblock (Termed "Dolly" in England).—The capblock is represented by spring \( K_1 \) in Fig. 1. The form of the stress-strain diagram (or the hysteresis loop) that is produced as the capblock is suddenly compressed and then allowed to reexpand, from information now available may be assumed to be as shown in Fig. 5.

Compression occurs along line \( AB \) whose slope is determined by the elastic constant \( K_1 \) of the capblock. Restitution occurs first along the line \( BD \) and then, because the capblock cannot transmit tension, is completed along line \( DA \), thus forming the hysteresis loop \( ABDA \). The slope of the line \( BD \) is automatically determined by the electronic computer so that

\[
\frac{\text{Area BCD}}{\text{Area ABC}} = \left(\frac{e_1}{1}\right)^2 = \frac{\text{Energy output}}{\text{Energy input}}
\]

where \( e_1 \) is the coefficient of restitution of the capblock. This is in accord with Newton's laws of impact.

Few tests have been made to determine the elastic characteristics of capblocks under impact conditions. The writer's former employer recently conducted a small number of tests by placing capblocks between two horizontally swinging rams weighing 4,800 lb each. The capblock was struck by one ram with about 15,000 ft-lb of energy, and the subsequent motion of both rams was recorded. A lead pellet was so mounted that it was squashed by the blow and thus measured the maximum compression of the capblock. From these measurements, the spring constant and the coefficient of restitution were determined mathematically.

It was found that the characteristics of a wood capblock vary during driving, but the tests led to the conclusion that in order to be on the conservative side in computing pile penetration per blow, a hardwood capblock with grain vertical, 6 in. in height originally, and with a horizontal area of \( A \) square inches, may be assumed to have the following characteristics:

- Spring constant, \( K = 20,000 \) A lb per in. of compression
- Coefficient of restitution, \( e = 50\% \)

The tests also indicated that a Micarta capblock (Nema Grade "C") 12 in. in height has the following characteristics (which do not vary much during driving):

- Spring constant, \( K = 45,000 \) A lb per in. of compression
- Coefficient of restitution, \( e = 80\% \)

Pile Cap or Follower or Helmet.—Like the ram of the hammer, the pile cap is ordinarily a short heavy rigid object that can be represented by a single weight without elasticity, such as \( W_2 \) of Fig. 1.

If the pile cap is long and slender, as is the case when it is to be used as a follower to drive the piles below ground or below water, then it may have to be represented by a number of weights and springs (as shown in Fig. 10(e), to be presented subsequently). In this case elastic constants must be computed. The elastic constants for any object of uniform cross section are computed by the well-known formula:

\[
K = \frac{A E}{l}
\]

where \( A \) is the sectional area in square inches, \( E \) denotes the modulus of elasticity in pounds per square inch, and \( l \) is the unit length, in inches, represented by a single spring.

Cushion Blocks (Called "Head Packing" in England).—In Fig. 1, springs \( K_2 \) to \( K_3 \), inclusive, represent the elasticity of the pile itself. However, if a precast concrete pile is being driven, a cushion block must be used under the pile cap \( W_2 \) so as to protect the concrete from shattering. In this case, spring \( K_2 \) in Fig. 1, would represent the cushion.

Fig. 5 applies to the cushion block as well as to the capblock, and computing methods for the cushion block and capblock are also alike.

Dynamic tests similar to those previously described for capblocks indicate that a wood cushion 4 in. thick composed of pine boards with grain horizontal, as used on top of a precut pile to distribute the blow evenly, may fairly be
assumed to have the following characteristics during driving, where $A$ is the area of the head of the pile:

Spring constant, $K = 3,480$ lb per in. of compression
Coefficient of restitution, $e = 50\%$

These characteristics vary widely during driving because the pile boards with grain horizontal become very hard and very elastic under continued pounding. The foregoing values are on the conservative side when used for computing pile penetration per blow.

Pile.—In Fig. 1, the ten springs with elastic constants $K_{2}$ to $K_{11}$, inclusive, represent the elasticity of the pile itself.

The experimental data available indicates small hysteresis loss in the pile itself, consequently, the springs representing the pile elasticity, such as $K_{2}$ to $K_{11}$ of Fig. 1, may, with negligible error, be considered to be perfectly elastic. (Eq. 27, which is given in the Appendix, makes possible the inclusion of hysteresis loss in the pile itself if in the future this appears to be worthwhile.)

In Fig. 1, springs $K_{3}$ to $K_{11}$ can transmit tension; but springs $K_{1}$ and $K_{2}$ cannot because the ram, the pile cap, and the pile are separate pieces. Therefore the electronic computer is programmed accordingly.

MATHEMATICS

The foregoing material outlines the problem and discusses the physical conditions that must be taken into account. The following gives the mathematics used for the numerical solution of the wave equation, as applied to pile driving.

In order to set up a program for the electronic computer, a number of equations and routines are required, as will be seen.

Notation.—The subscript $m$ denotes the general case; thus $W_{m}$ denotes any weight as in Fig. 1. For instance $W_{m}$ might denote $W_{8}$, in which case $K_{m}$ would denote $K_{8}$ and $R_{m}$ would denote $R_{8}$, etc. Thus the letter $m$ denotes position in space. The letter $n$ is used to denote position in time.

The capital letters $C$, $D$, $F$, $R$, $V$, and $Z$, refer to instantaneous values for compression, displacement, force, resistance, velocity, and accelerating force, calculated for any time interval $n$. The corresponding small letters refer to corresponding values in the preceding time interval $n-1$. The notation $d^*$ refers to a displacement value in interval $n-2$ (two intervals back). Compressions and displacements are totals up to and including the time interval specified:

$A =$ cross-sectional area, in square inches;

$C =$ spring compression in time interval $n$, in inches;

c = spring compression in time interval $n-1$, in inches;

$D =$ displacement in time interval $n$, in inches;

d = displacement in time interval $n-1$, in inches;

d* = displacement in time interval $n-2$, in inches;

$D' =$ ground plastic displacement in time interval $n$, in inches;

d' = ground plastic displacement in time interval $n-1$, in inches;

$E =$ modulus of elasticity, in pounds per square inch;

e = coefficient of restitution;

$F =$ force exerted by spring in time interval $n$, in pounds;

$g =$ acceleration due to gravity, in feet per second per second;

$J =$ damping constant applicable to resistance at point of pile (R12 of Fig. 1);

$J' =$ damping constant applicable to resistance at side of pile (R9 to R11 of Fig. 1);

$K =$ spring constant, in pounds per inch;

$K' =$ spring constant applicable to ground, in pounds per inch;

$l =$ unit length of pile, in inches;

$m =$ subscript denoting the general case;

$n =$ time interval for which calculations are being made;

$p =$ subscript denoting "at point of pile";

$Q =$ quake or maximum elastic ground deformation, in inches;

$R =$ resistance in time interval $n$, in pounds;

$R_{u}$ = total ultimate ground resistance to driving, in pounds;

$R_{um}$ = portion of $R_{u}$ applicable to weight $W_{m}$;

$s =$ permanent set of pile per blow, in inches;

$T_{m}$ = critical time interval for spring $m$, in seconds;

$\Delta t =$ time interval used for calculation, in seconds;

$V =$ velocity in time interval $n$, in feet per second;

$V =$ velocity in time interval $n-1$, in feet per second;

$W =$ weight, in pounds; and

$Z =$ accelerating force in time interval $n$, in pounds.

Equations.—In a previous paper the writer derived the following five basic equations from the elementary laws of physics. They are based on the assumption that all springs are perfectly elastic, and that the pile is represented typically as shown in Fig. 1:

\[
D_{m} = d_{m} + v_{m} (12 \Delta t) \quad \dot{\ldots} \quad (4)
\]

\[
C_{m} = D_{m} - D_{m+1} \quad \dot{\ldots} \quad (5)
\]

\[
F_{m} = C_{m} K_{m} \quad \dot{\ldots} \quad (6)
\]

\[
Z_{m} = F_{m} - 1 - F_{m} - R_{m} \quad \dot{\ldots} \quad (7)
\]

Referring to Fig. 1, the velocity of any particular weight in any particular time interval, produces a displacement in the next time interval as given by Eq. 4. The displacements of two adjacent weights produce a compression in the spring between them as given by Eq. 5. This spring compression results in a spring force as given by Eq. 6. The two spring forces and the resistance acting on one particular weight produce a net force as given by Eq. 7. This net force accelerates or decelerates the weight and produces a new velocity as given by Eq. 8. This new velocity, in turn, produces a new displacement in the next succeeding time interval, etc. The process is repeated for each weight and each spring in each time interval, until all downward velocity is lost.

Subsequently these equations will be modified so as to provide for inelastic and plastic characteristics of the capblock, cushion block, and the ground.

If, by methods well known to mathematicians,\(^7\) the wave equation, with resistance included, is converted into a difference equation suitable for numerical computation, the following expression is obtained:

\[
D_m = 2d_m - d^* + \frac{12g\Delta t}{W_m} \left[ (d_{m-1} - d_m)K_{m-1} - (d_m - d_{m+1})K_m - R_m \right].
\]

Eq. 9 can also be obtained by combining Eqs. 4 through 8 into a single equation. This shows that these five basic equations are equivalent to the wave equation for purposes of numerical computation.

Eqs. 4 through 8 may be combined in numerous ways. For instance, \(Z_m\) can be eliminated by combining Eqs. 7 and 8 thus:

\[
V_m = v_m + \left( F_{m-1} - F_m - R_m \right) \frac{\Delta t g}{W_m} \quad \ldots \quad (10)
\]

Similarly, \(C_m\) can be eliminated by combining Eqs. 5 and 6 thus:

\[
F_m = \left( D_m - D_{m+1} \right) K_m \quad \ldots \quad (11)
\]

The ultimate ground resistance \(R_u\) may be distributed throughout the length of the pile in any way desired by writing

\[
R_u = R_{u1} + R_{u2} + R_{u3} \ldots \quad \ldots \quad (12)
\]

Any of these individual ultimate resistances may be denoted by the general expression \(R_{um}\) (Fig. 1).

In connection with Fig. 4, it was explained that the ground is assumed to compress elastically a distance \(Q\), after which the ground fails plastically at constant resistance, which for the general case of weight \(W_m\) would be called

\[R_{um}.\]

For computation this may be represented diagrammatically as in Fig. 6. In Fig. 6, springs \(K_m'\) and \(K_m''\) are introduced to represent ground elasticity. Spring \(K_m'\), the general case, must compress a distance \(Q\) in order to reach the individual ultimate ground resistance \(R_{um}\); therefore, to compute the spring constant for the ground we may write:

\[
K_m' = \frac{R_{um}}{Q} \quad \ldots \quad \ldots \quad (13)
\]

In Fig. 6, \(D_m\) is the displacement of weight \(W_m\) from its original position, as computed by Eq. 4, and \(D_m'\) is the measure of the plastic ground failure permitting movement in excess of \(Q\). The amount that ground spring \(K_m''\) is compressed may be seen from Fig. 6, to be equal to \(D_m - D_m'\). From this we may write an equation for \(R_m\) similar to Eq. 11:

\[
R_m = (D_m - D_m') K_m'' \quad \ldots \quad \ldots \quad (14)
\]

In order to include viscous damping as already discussed under the heading "General: Soil Mechanics," Eq. 14 is modified to include a damping constant \(J\) or \(J'\) multiplied by the instantaneous velocity. For frictional resistance alongside the pile, Eq. 14 thus becomes:

\[
R_m = (D_m - D_m') K_m'' (1 + J' v_m) \quad \ldots \quad \ldots \quad (15)
\]

When applied specifically to resistance at the point of the pile this becomes:

\[
R_p = (D_p - D_p') K_p' (1 + J v_p) \quad \ldots \quad \ldots \quad (16)
\]

Routines.—Evaluating ground displacements \(D_m\) and \(D_m'\) as used in Eqs. 15 and 16 involves the use of computer routines rather than equations.

Routine #1.—The displacement \(D_m\) remains constant (starting at zero) unless changed by one of the two following conditions:

1. \(D_m\) cannot be less than \(D_m - Q\)
2. \(D_m\) cannot be more than \(D_m + Q\)

The computer makes these comparisons in each time interval and adjusts the value of \(D_m\) accordingly:

Remarks.—Routine #1 applies only to friction alongside the pile, and insures that the compression or expansion of spring \(K_m''\) cannot exceed \(Q\), the maximum elastic ground deformation. When the upper part of the pile rebounds after the blow, the motion reverses itself; consequently, routine #1 provides for plastic movement \(D_m'\) either in the normal downward direction or in an upward direction resulting from pile rebound. This routine is illustrated in Fig. 7.

Routine #2.—The displacement \(D_p\) remains constant (starting at zero) unless changed by the following condition:

\(D_p\) cannot be less than \(D_p - Q\)

The computer makes this comparison in each time interval and adjusts the value of \(D_p\) accordingly.

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Remarks.—Routine #2 applies only to the point of the pile. It is similar to routine #1 except that only downward or positive plastic ground movement $D_p$ is considered. The maximum value of $D_p$ is the "permanent set" of the pile, which is called $s$, as shown in Fig. 4. Therefore, it would be illogical to consider reversal of plastic failure of the ground as was done in routine #1. This routine is illustrated in Fig. 8.

Routine #3.—This routine provides for inelasticity in spring $K_1$ by taking account of its coefficient of restitution $e_1$, and involves the alternate use of Eqs. 17 and 18 as follows:

$$F_1 = C_1 K_1$$

for compression. This equation is identified with the number 17 merely as a matter of convenience. Actually it is nothing but Eq. 6 applied to spring $K_1$.

For restitution

$$F_1 = \left[ \frac{K_1}{(e_1)^2} \right] C_1 \left[ \frac{1}{(e_1)^2} - 1 \right] K_1 C_{1 \text{ max}}$$

in which $e_1$ is the coefficient of restitution for spring $K_1$ and $C_{1 \text{ max}}$ is the maximum value of $C_1$, used as a constant.

It should be noted that if $e_1 = 1.00$, then Eqs. 17 and 18 become identical and equivalent to Eq. 6.

Eq. 18 is used for restitution. Its derivation is based on the relationships previously explained in connection with Fig. 5, and is not difficult.

Routine #3 is based on a stress-strain (or force-compression) diagram as given in Fig. 9. Letters (a), (b), etc., in Fig. 9, correspond to the correspondingly numbered steps in the routine, as given subsequently.

The actual routine is as follows:

(a) Use Eq. 17 until $C_1 - c_1$ becomes negative. This last value of $c_1$ is thereafter treated as a constant called $C_{1 \text{ max}}$ for use in Eq. 18.

(b) Use Eq. 18 instead of Eq. 17.

(c) Sometimes recompression occurs; nevertheless, continue using Eq. 18 until $C_1 - C_{1 \text{ max}}$ becomes positive.

(d) Then again use Eq. 17 until $C_1 - c_1$ again becomes negative. This gives a new value for $C_{1 \text{ max}}$ equal to the latest value of $c_1$.

(e) Then use Eq. 18 with the new value of $C_{1 \text{ max}}$ as a constant. (If additional recompressions occur, steps (c), (d), and (e) are repeated).

(f) $F_1$ can never be less than zero, that is, can never be negative, because the ram is always a separate piece.

Routine #4.—This routine provides for inelasticity in spring $K_2$ by taking account of its coefficient of restitution $e_2$. It uses two equations corresponding to Eqs. 17 and 18, thus:

$$F_2 = C_2 K_2$$
for compression, and
\[ F_2 = \left[ \frac{K_2}{e_2} \right] C_2 - \left[ \frac{1}{(e_2)^2} - 1 \right] K_2 C_{2\max} \]  \hspace{0.5cm} (20)

for restitution.

Note that if the coefficient of restitution \( e_2 = 1.00 \) Eqs. 19 and 20 become identical and equivalent to Eq. 6.

The routine is the same as routine #3, with subscripts changed from 1 to 2, with the one exception of item (f) of routine 3. For routine #4, item (f) reads as follows:

(f) Alternate No. 1: \( F_2 \) cannot be less than zero (that is, cannot be negative).

(f) Alternate No. 2: \( F_2 \) can be less than zero (that is, can be negative).

The proper alternate must be specified. Alternate No. 1 applies if \( K_2 \) cannot transmit tension. Alternate No. 2 applies if \( K_2 \) can transmit tension.

PROGRAM FOR ELECTRONIC DIGITAL COMPUTER

By combining the foregoing formulas and routines a complete program for the computer can be prepared.

The calculation starts with a specified ram velocity \( V_1 \) at the "beginning of impact" denoted as time interval \( 0 \). All other variables except \( V_1 \) start with a value of zero in time interval \( 0 \), and remain at zero until modified by one of the following items bearing Roman numerals, which comprise the computer program. These items are listed in the order in which they are calculated by the computer in each and every time interval successively. Time interval #1 is the first time interval for which computations are made by the electronic computer.

Computer Program #1 For Piles With All Resistance At The Point Of The Pile

I Compute \( D_1 \) to \( D_p \) by means of Eq. 4.
II Compute \( D'_p \) by means of routine #2.
III Compute \( R_p \) by means of Eq. 16.
IV Compute \( C_1 \) to \( C_{p-1} \) by means of Eq. 5.
V Compute \( F_1 \) by means of routine #3.
VI Compute \( F_2 \) by means of routine #4.
VII Compute \( F_3 \) to \( F_{p-1} \) by means of Eq. 6.
VIII Compute \( V_1 \) to \( V_p \) by means of Eq. 10.

Computer Program #2 For Piles With Side Friction As Well As Point Resistance

I to VII inclusive, same as above.

VIII Compute \( D'_3 \) to \( D'_{p-1} \) by means of routine #1.
IX Compute \( R_3 \) to \( R_{p-1} \) by means of Eq. 15.
X Compute \( V_1 \) to \( V_p \) by means of Eq. 10.

In the above programs Eq. 10 has been used instead of Eqs. 7 and 8 because the accelerating force \( Z \) is seldom of interest and the use of Eq. 10 saves computer time.

FIG. 8.—ILLUSTRATING ROUTINE NUMBER 2

Note: \( D'_p \) lags behind \( D_p \) on the way down by distance \( Q \). \( D'_p \) represents permanent set, therefore never decreases.

FIG. 9.—STRESS-STRAIN DIAGRAM FOR SPRING \( K_1 \)

The computer should be programmed to stop automatically when the following two conditions are reached:

(1) \( D'_p \) - \( d'_p \) equals zero.
(2) \( V_1 \) to \( V_p \) inclusive, are all simultaneously negative or equal to zero.

This will ordinarily stop the calculation very soon after \( D'_p \) reaches a maximum and the point of the pile begins to rebound as indicated in Fig. 8. By this time the major driving force has been expended and only secondary or residual forces are acting. These are of little interest and are difficult to analyze accurately.
RECOMMENDATIONS AND COMMENTS

The mathematical method explained herein was originally developed as a means of computing stresses during driving in capblocks, pile caps, piles, and pile-driving cores or mandrels, in order to insure proper design. Previously all these parts had been designed by trial and error.

During the past 12 yr this mathematical method has been used for over 1,000 calculations, and has been gradually improved and extended to cover pile driving in general. It is now being presented with the hope that it will be useful in determining stresses during driving, and especially stresses in precast concrete piles, so that these piles may be provided with correct reinforcement and correct driving procedures in order to prevent or eliminate cracking or breakage during driving. It is also hoped that in due time the results of many calculations will be correlated with the results of pile-load tests carried to failure, and also with the rapidly increasing knowledge of soil mechanics.

In order to make calculations at all, values must be assigned to certain constants that describe soil action during driving. Up to the present time no instrumented field experiments have been performed to determine these constants accurately. From experience gained through working extensively with this problem, and as a result of making a limited number of comparisons with load tests carried to failure, the writer has personally concluded that certain values are accurate enough for practical use until such time as more accurate values become available. These values are

\[
Q = 0.10 \text{ in.} \\
J = 0.15 \\
J' = 0.05
\]

The writer is emboldened to make the foregoing recommendations (which may easily be criticized as rash) because he has found from the many calculations already performed that the numerical wave equation solution is not "sensitive," that is, a small change in the value assigned to any constant will produce a smaller change in the calculated results. He therefore believes that the results of calculations are meaningful and worthwhile even though the constants used are not as accurate as might be desired.

To determine a suitable value for \( R_u \) to be used in a calculation, the working load to be placed on the pile must be multiplied by a factor such as 2 or 3. However, if soil investigation indicates that the soil is of a type that "sets up" or one that "relaxes" after driving, this factor should be modified accordingly. The numerical wave equation calculation tells what happens during the actual driving of the pile; it cannot predict what will happen a week or a year later.

For this information soil mechanics must be consulted.

Modern large electronic computers can complete a calculation such as described in this paper, in a matter of seconds. It is therefore practical to make a large number of calculations and to tabulate (or plot) the results for future reference. Such a tabulation would cover the types of piles and hammers in common use. Special calculations would then need to be made only for unusual types.

The following conclusions come partly from the results of numerical wave equation calculations, but primarily from many years of practical pile-driving experience.

PILE DRIVING

CONCLUSIONS

1. A numerical method suitable for use on modern electronic computers is now available that permits calculation of pile-driving action under any specified set of conditions, and gives permanent set per blow as well as instantaneous stresses, displacements, and velocities.

2. The knowledge of soil mechanics is incomplete, especially the knowledge of soil mechanics under pile-driving action. This offers a fertile field for future investigation, especially now that a mathematical method of calculating driving action is available for checking and analyzing field test results.

3. The knowledge of the physical action of capblocks and cushion blocks under driving conditions could also be improved by dynamic testing. Only a few tests have been made.

4. The numerical wave equation calculation can be used to determine the driving characteristics of various types of piles and hammers. It can also be used to determine the range of application throughout which any particular pile-driving formula may be considered reasonably accurate.

5. It is well known that precast concrete piles are sometimes cracked or broken due to excessive compressive stresses caused by the hammer blow. It is not equally well known that in certain cases excessive tensile stresses may also result from the hammer blow, especially if the pile is long. A wave equation analysis can be used to determine correct reinforcement and driving procedures to help eliminate these troubles.

6. Many engineers assume that at final penetration, a light pile is easier to drive than a heavy pile of the same overall dimensions. Actually this assumption is correct only for light loading. When loads are heavy and consequently resistance is high, a heavy pile is usually easier to drive than a light pile, because it is stiff and strong and thus is better able to carry large forces down into the ground.

APPENDIX

This appendix is presented primarily for those who actually make calculations, and gives practical details that will be found helpful.

In the writer's paper, a recommendation was made that, in representing the pile, the weight of each unit length should be concentrated at the center. Experience since that paper was written has shown that a preferable procedure is to concentrate the weight of each unit length at the end away from the place where impact occurs, as was done in Fig. 1. This procedure improves accuracy and permits any combination of pile, hammer, etc., to be represented logically as shown in Fig. 10.

The elements shown in Fig. 10 are combined in the following typical ways to form single diagrams similar to Fig. 1:

1. Ram, capblock, pile cap and pile.
2. Ram, capblock and core or mandrel.
3. Ram and pile.
4. Ram, capblock, long follower and pile.
5. Ram, capblock, pile cap, cushion and pile. In this case the spring rep-
representing the cushion must be combined with the top spring of the pile by means of Kirchhoff’s law

\[ \frac{1}{K} = \frac{1}{k_1} + \frac{1}{k_2} \] \hspace{1cm} (21)

---

**Fig. 10.—REPRESENTATION OF PILE AND DRIVING ELEMENTS**

6. Long ram, capblock, pile cap and pile. In this case the spring representing the capblock must be combined with the bottom spring of the ram. (See #5 above).

7. Ram, pile cap or anvil and pile. In this case combine the weights representing the ram and the pile cap, and assign a velocity at impact so that the momentum of the combined weight equals the momentum of the ram at impact.

The best time interval \( \Delta t \), for use in making a pile calculation, may be defined as the largest interval that will produce a completely stable calculation.

---

A simple rule can be given that will cover every possible condition, but the writer has discussed this question and has presented two formulas for the “critical” time interval based on the velocity of travel of the stress-wave as follows:

\[ T_m = \frac{1}{1.9648} \sqrt{\frac{W_m + 1}{K_m}} \] \hspace{1cm} (22)

and

\[ T_m = 0.0508 \sqrt{\frac{W}{K}} \] \hspace{1cm} (23)

The minimum value of \( T_m \) that can be obtained by using these formulas is the “critical” one.

The time interval \( \Delta t \) should be about half of this critical value so as to prevent instability from arising due to other factors not included in the above formulas, such as the effects produced by coefficients of restitution \( e \), ground quake \( Q \) and damping constants \( J \) and \( J' \). As a general rule, the following values for \( \Delta t \) will be found to be satisfactory for use with piles divided into 8 ft or 10 ft unit lengths:

- For steel piles: 1/4,000 sec
- For wood piles: 1/4,000 sec
- For concrete piles: 1/3,000 sec

If shorter unit lengths are used, the time interval \( \Delta t \) should be reduced proportionately.

An interesting and instructive combination of figures for use with a steel pile of uniform section is to take \( l = 100 \) in. (8.33 ft), \( \Delta t = 1/4,000 \) sec, and \( E = 29,300,000 \) psi. Sound or stress waves travel at a speed equal to \( \sqrt{E/\rho} \) where \( \rho \) is the mass per unit volume. Steel weighs 0.283 lb per cu in. and the velocity of sound is 16,660 fps. Thus sound or stress waves will traverse an 8.33 ft unit length in 1/2,000 sec. If 1/4,000 sec is used for \( \Delta t \), forces will progress from one pile spring to the next (such as from spring \( K_4 \) to \( K_5 \) of Fig. 1) every second time interval. Similarly velocities will progress from one pile weight to the next (such as \( W_4 \) to \( W_5 \) of Fig. 1) every second time interval. This may be observed from the numerical wave equation calculation, though the figures will not come out absolutely exact. However, when the calculation has been carried on long enough so that the stress wave is reflected from the point of the pile, the wave action may become harder to observe from the calculated figures, though sometimes the travel of the reflected wave up the pile is easy to follow. If ground friction alongside the pile is included in the calculation, the sound or stress wave will travel at a slower speed.

If the time interval used is too large, instability in the calculation may result. Usually this will force the computer to handle such large numbers that it will give warning. A good plan is to program the computer so that it will give warning if the value of \( V_2 \) or \( V_p \), in any time interval, exceeds twice the velocity of the ram at the moment of impact (that is, twice \( V_1 \) in time interval 0). This will usually detect instability promptly. It is also a good plan to program the computer to stop the calculation after a maximum of 300 time intervals. This will eliminate any possibility of the computer running indefinitely if the regular program does not stop the calculation at the correct point, as in Fig. 8.
As a check, more than one calculation should be made for any particular pile and hammer combination using varying values of \( R_u \). If four or five such calculations are made, the calculated values of \( s \) should plot as a smooth curve. If they do not, it may be concluded that instability was present, or that some error has been made in the calculation. Other methods of checking have been given by the writer.  

In rare cases the capblock spring \( K_1 \) may be very stiff and the weight \( W_2 \) may be very light. This combination tends to produce instability, and if tested according to Eqs. 22 and 23, may call for an excessively small time interval. From a practical standpoint it may be preferable to arbitrarily soften spring \( K_1 \) somewhat, or to arbitrarily increase the weight of \( W_2 \). Either or both of these changes will permit use of a normal time interval such as 1/4,000 sec, will have only a small effect, and will give results on the safe or conservative side.

It should be noted that if the hammer ram is long and is represented by a number of weights and springs as shown in Fig. 10 (b), then all these weights will have equal initial velocities \( V_1, V_2, V_3 \), etc., and the computer must be specially programmed to accept all these velocities as input data.

It should also be noted that the methods described herein may be used to analyze the driving of a pile in which the ram strikes its blow part way down in the pile, or all the way down at the point. Fig. 11 shows, diagrammatically, a pipe pile being driven by a drop hammer operating inside the pile and striking on the point. All equations already given apply to Fig. 11 as well as to Fig. 1 except that the compression of spring \( K_1 \) must be computed by the equation

\[
C_1 = D_1 - D_{12} 
\]

(24)

The accelerating force for weight \( W_{12} \) must be computed by the equation

\[
Z_{12} = F_1 + F_{11} - R_{12} 
\]

(25)

and the accelerating force for weight \( W_2 \) by the equation

\[
Z_2 = -F_2 = R_2 
\]

(26)

These changes are obvious from Fig. 11, and of course require special computer programming.

It is of great interest to note that driving on the point of the pile in this way has been found to be no more effective than driving on the upper end of the pipe. The explanation appears to be that driving on the point of the pile tends to produce high point velocity, which causes high temporary resistance because of soil viscosity. This illustrates the necessity of introducing the damping constant \( J \).

Some types of ground furnish side frictional resistance during driving, but add little or nothing to the permanent bearing capacity of the pile. In this case side resistances may be used in the calculation but ignored in determining the bearing capacity of the pile. Thus in Eq. 12 \( R_{12} \), \( R_{14} \), etc., would be given values, but only \( R_{up} \) would be considered to be permanent bearing capacity.

The writer has also presented formulas for computing \( K \)-values for tapered piles.

It is usual to ignore gravity forces because the pile-bearing capacity wanted is the bearing capacity that the pile has in addition to its own weight. Anyone who wishes to include gravity will find methods of doing so discussed elsewhere by the writer.

Illustrative Problem.—Let Fig. 1 represent a steel pipe or H pile 100 ft long with 15.88 sq in. of area, weighing 53 lb per ft, driven by a #1 Vulcan hammer having a 5,000 lb ram falling 3 ft, using a 700 lb pile cap and a hardwood cap-block 6 in. thick when new and 11-1/2 in. in diameter. A special pile point is specified weighing 100 lb.

Assuming that hammer efficiency is 80%, ground quake Q is 0.1 in., and damping constant \( J = 0.15 \), determine the permanent set per blow when driving against an ultimate ground resistance \( R_u \) of 200,000 lb concentrated at the point of the pile.
Step No. 1, Choice of Unit Length.—A 10 ft unit length will be adopted. Experience has shown this unit length to be generally satisfactory.

Step No. 2, Choice of Time Interval.—As a general rule, a time interval \( \Delta t = 1/4,000 \) sec will produce a stable calculation using 10 ft unit lengths, and this will be used here.

Step No. 3, Determination of Weights \( W_m \):—
- \( W_1 = \) the hammer ram = 5,000 lb;
- \( W_2 = \) the pile cap = 700 lb;
- \( W_3 \) to \( W_{11} = 10 \) ft unit lengths of pile at 53 lb per ft = 530 lb each;
- \( W_{12} \) includes a special pile point weighing 100 lb, therefore \( W_{12} = 630 \) lb.

Step No. 4, Determination of Spring Constants \( K_m \) and \( K_{m'} \):—The term \( K_1 \) represents the capblock. This has approximately 100 sq in. of area, and it was stated under the heading "General: Physics: Capblock," that the spring constant for such a capblock = 20,000 A, therefore \( K_1 = 20,000 \times 100 = 2,000,000 \) lb per in. of compression.

The terms \( K_2 \) to \( K_{11} \) inclusive, each represent the elasticity of a 10 ft unit length of pile. This is computed from the formula:

\[
K = \frac{A E}{1} = \frac{15.58 \times 30,000,000}{10 \times 12} = 3,900,000
\]

Since all resistance is assumed to be at the point of the pile, all values of \( K_m \) except \( K_1 \) are equal to zero. In this case \( K_p \) is \( K_{12} \). From Eq. 13 we get \( K_{12} = 2,000,000 \) lb per in. of compression.

Step No. 5, Determination of Coefficient of Restitution.—The pile has been assumed to be perfectly elastic as discussed under the heading "General: Physics: Pile," consequently, the computer program requires coefficients of restitution only for the first two springs \( K_1 \) and \( K_2 \).

Spring \( K_1 \) represents the wood capblock for which a value of \( e_1 = 0.50 \) is suitable.

Spring \( K_2 \) in this case represents part of the pile, therefore, \( e_2 = 1.00 \).

Step No. 6, Determination of Impact Velocity.—The rated energy of the hammer is 5,000 \( \times 3 = 15,000 \) ft-lb. The impact velocity is computed by using Eq. 1:

\[
\text{Impact velocity, in fps} = \sqrt{\frac{15,000 \times 0.80 \times 64.4}{5,000}} = 12.4 \text{ fps.}
\]

Step No. 7, Decide Whether or Not Spring \( K_2 \) Can Transmit Tension.—The term \( W_2 \) represents the pile cap which merely rests loosely on top of the pile, therefore spring \( K_2 \) cannot transmit tension, therefore \( F_2 \) can never have a negative value and its minimum value will be \( 0 \).

Step No. 8, Summary.—The foregoing values may be summarized for computer input as follows:

- \( \Delta t = 1/4,000 \)
- \( W_1 = 5,000, \ W_2 = 700, \ W_3 \) to \( W_{11} = 530 \) each, \( W_{12} = 630 \)
- \( K_1 = 2,000,000, \ K_2 \) to \( K_{11} = 3,900,000 \) each, \( K_{12} = 2,000,000 \)
- \( e_1 = 0.50, \ e_2 = 1.00 \)

Initial value of \( V_1 = 12.4 \)
- \( Q = 0.1 \)
- \( J = 0.15 \)

\( F_2 \) can be equal to or greater than zero, but can never be less than zero.

If computer program #1 is being used the foregoing constitutes the required input data. If computer program #2 is being used, the following must also be specified:

- \( K'_{1} \) to \( K'_{11} = 0 \)
- \( J' = 0 \)

Step No. 9, Computation.—The values listed in step 6 are used as computer input for a computer that has been programmed in accordance with Program #1 or #2.

This has already been done for this particular problem. The permanent set was computed to be 0.20311 in. per blow. (5 blows per in.) The computer was programmed to print maximum compressive and tensile forces in the pile. These were found to be as follows:

<table>
<thead>
<tr>
<th>Maximum Forces</th>
<th>Compression</th>
<th>Tension</th>
</tr>
</thead>
<tbody>
<tr>
<td>At head of pile (( F_2 ))</td>
<td>290,000 lb</td>
<td>18,600 psi</td>
</tr>
<tr>
<td>At middle of pile (( F_6 ))</td>
<td>300,000 lb</td>
<td>19,250 psi</td>
</tr>
<tr>
<td>At point of pile (( F_{11} ))</td>
<td>405,000 lb</td>
<td>26,000 psi</td>
</tr>
</tbody>
</table>

Forces such as these are primarily of interest as a means of determining unit stresses during driving in the case of precast and prestressed concrete piles.

In the above example tension did not occur because \( R_u = 100 \) tons which is a moderately high value. Tension in piles occurs primarily during the early stages of driving when \( R_u \) is quite small.

The compressive stress at the point of the pile is higher than at the head because the fairly high value of \( R_u \) produces a reflected compressive wave which travels back up to the head of the pile. This upward wave overlaps the original downward traveling compressive wave and thus increases the stress at the point of the pile.

The calculated displacements \( D_1 \) to \( D_{12} \), which apply to weights \( W_1 \) to \( W_{12} \), as well as the plastic ground displacements \( D_p \) are plotted in Fig. 12. Each dot on each curve represents a computed point. Fig. 12 gives a lot of information and is worthy of study.

For The Future.—Future investigators with more complete experimental data available may consider it worthwhile to include hysteresis (internal damping) in the pile as part of the calculation. This can be done by modifying Eq. to read as follows:

\[
F_m = C_m K_m + B \left( \frac{C_m - C_m}{12 \Delta t} \right)
\]
where $B$ is a hysteresis or internal damping constant. A small value (such as 0.0002) should be assigned to $B$ so as to produce a narrow hysteresis loop. Actually, the "loop" is a sort of spiral starting at the origin and returning to the origin when both $C_m$ and $C_m' - C_m$ become equal to zero.

The expression \[ \frac{C_m' - C_m}{12 \Delta t} \] in Eq. 27 is the instantaneous rate of spring compression. This corresponds in a general way to $v_m$ and $v_p$ in Eqs. 15, 16, 28, and 29.

A recommendation has been given previously to the effect that the calculation be stopped soon after maximum pile point penetration is reached as indicated in Fig. 8 and actually done in Fig. 12. Future investigators may want to continue beyond this point. If so, they should take note of the fact that Eqs. 15 and 16 give no damping at all when $(D_m - D_m')$ becomes zero, even though the velocity may be considerable. To overcome this difficulty, Eqs. 15 and 16 can be used until $(D_m - D_m')$ or $(D_p - D_p')$ is first equal to $Q$, which corresponds to point A of Fig. 4. From this point on the following equations may be used instead of Eqs. 15 and 16, respectively:

\[ R_m = (D_m - D_m') K_m' + J K_m' Q v_m \quad \cdots \cdots \cdots \cdots (28) \]

and

\[ R_p = (D_p - D_p') K_p' + J K_p' Q v_p \quad \cdots \cdots \cdots \cdots (29) \]

These two equations produce damping at all times except when $v_m$ (or $v_p$) is equal to zero. Changing equations in this way is similar to the use of two different equations in routines #3 and #4.

It is also possible that Eq. 27 may in the future be found suitable for use with the capblock and cushion block, and thus replace routines #3 and #4. Before this can be done, suitable values for $B$ in Eq. 27 would have to be determined experimentally.

**DISCUSSION**

L. O. SODERBERG, 8 M. ASCE.—Mr. Smith has proposed a method of analyzing pile driving that goes far beyond a "pile driving formula." He offers a rational approach for studying many of the subtleties of piles that have been treated empirically in the past.

As a great deal of numerical manipulation is required to attain a solution, any simplification of these procedures will help to make the method more useful. By treating the problem as one of finite differences, significant simplifications can be attained with no loss of accuracy.

Mr. Smith can treat a tapered pile with a large variety of end conditions. His basic assumption is essentially one dimensional wave propagation where

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8 Research Engr., Raymond International Inc., New York, N. Y.
small variations in the pile cross sectional area are permitted. The partial
differential equation describing this phenomenon is

$$E \frac{\partial^2 u}{\partial x^2} + \frac{\partial A}{\partial x} \frac{\partial u}{\partial x} - \rho A_x \frac{\partial^2 y}{\partial t^2} \frac{\partial y}{\partial x} = 0$$

in which $A_x$ represents the cross sectional area of the pile at point $x$, $u$ is the
displacement of the pile at point $x$, $R$ is the total skin friction acting on element
$dx$, and $\rho$ is the mass density of the pile.

Eq. 30 can be approximated directly by the technique of finite differences: 

$$D_m = 2 d_m (1 - \phi) d_{m+1} \left(1 + \frac{A_{m+1} - A_{m-1}}{4 A_m}\right)$$

$$+ \phi d_{m-1} \left(1 - \frac{A_{m+1} - A_{m-1}}{4 A_m}\right) - \frac{R}{A_m E} d_m^*$$

This notation is Mr. Smith's with the exception that

$$\phi = \frac{E \Delta t^2}{\rho \delta^2}$$

If the cross sectional area of the pile is constant Eq. 31 is exactly Eq. 9 with
the constants that determine $\phi$ factored out. If the areas vary, these two equa-

![Table 1]

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Displacements in .001 in/per time unit = 1/4000 sec.

The first ten time intervals of Mr. Smith's illustrative problem are shown in Table 1. This computation treats only displacements and the initial conditions must be entered as such. At time 0 all displacements are 0, and at time 1 all displacements are still 0 with the exception of the hammer. The hammer has moved a distance equal to its initial velocity times one time interval. Next, the relationships Eq. 33 through Eq. 36 are used working up from left to right one time interval at a step.

The numerical work is essentially a weighted averaging process. As one be seen from Table 1 the points on Fig. 12 are produced accurately with a slide rule.

The relationships developed here are for capblock compression and point displacement up to 10 in. For capblock restitution and displacements in excess of .10 in. new relationships for the hammer, pile cap, and pile tip must be used.

These simplifications of Mr. Smith's basic method have been used successfully both in hand computations and on a small scale digital computer and have been found to reduce the computational work by as much as a factor of 4.

MARVIN GATES, 10 M. ASCE.—The application to pile driving of the wave equation, developed almost one hundred years ago by St. Venant 11 and later by Boussinesq, 12 is proposed by Mr. Smith in this paper as well as in several others of his writings. 13,14,15,16 Mr. Smith is deserving of the Society's appreciation for his untiring efforts to acquaint the profession with a tool which, though little used now, may become the ultimate rationale in the area of pile driving analysis.

D. V. Issacs 17 first proposed the application to pile driving of the wave equation; followed by W. H. Glanville 18 et al., seven years later in 1938. A. E. Cummings in 1940, 19 and again in 1941 20 reported on the works of the aforementioned investigators but made little contribution of his own to the furtherance of either the theory or its applications to pile driving. Ten years elapsed when the wave equation was once again proposed to the profession by Mr. Smith, 19 and ever since he has continued untiringly to update his writings.

However, in all of Mr. Smith's efforts, he implies that the wave equation can be used to solve for the bearing capacity as well as the driving stresses. This sweeping application was objected to by this writer as well as others. 21

Mr. Smith subsequently replied to these objections, 22,23 Recently Mr. Smith announced the publication of this paper and an improvement on his earlier article. At the same time, specific reference was again made to bearing capacity.

It is important to note that the investigators preceding Mr. Smith unanimously cautioned against the use of the wave equation as a means of determining bearing capacity. Rather, all concurred that its application should be restricted to finding the driving stresses in piles. Some investigators went so far as to limit its application to certain types of piles under particular conditions of driving. In view of these facts, it is suggested that Mr. Smith make a definitive statement concerning the limits of application of his presentation. This is necessary to remove the haze precipitated by certain apparently conflicting statements concerning the use of this method. Mr. Smith's qualification of his previous references to bearing capacity, by mentioning in this paper soils which may either relax or set-up is no clarification at all. Because again, by implication, the wave equation will give the true bearing capacity for piles driven in soils which return to their natural state after, or remain unchanged during the pile driving operation.

There are many objections to the use of a dynamic formula in general and the wave equation in particular, to determine the bearing capacity of piles. It is sufficient here, to avoid repetition, to cite several authorities in substantiation of this statement. 25 In an important treatise on this subject, 26 the question of the application to bearing capacity of dynamic type formulas is deftly summed up as follows:

It is true that some of the (dynamic) formulas at times give results approximately correct. This is because they empirically apply to certain piles and conditions of driving.

The principal objection to the use of any dynamic formula is, of course, the attempt to equate an instantaneous (about 0.02 sec) kinetic load to a long term static load. On the other hand, static formulas lack general applicability on account of the wealth of data relating to the physical properties of the soil which is required.

However, empirical equations are frequently employed when the number of variables and their inter-relationships are not all known. Chellis 27 lists several empirical pile driving formulas. More recently, this discussor proposed an empirical relationship 28 based on a limited statistical investigation of piles loaded to failure. Although containing but two simple parameters, this relationship gives more consistently accurate results than the most complex

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12 Application des Potentiels, 1855, p. 508.
19 Journal, Boston Society of Civil Engineers, 1940.
20 Proceedings, ASCE, 1941.
22 Engineering News Record, April 24, 1958, p. 8
23 Engineering News Record, June 19, 1958, p. 10
24 Engineering News Record, September 22, 1960, p. 32.
26 Pile Foundations and Pile Structures, ASCE Manual No. 27.
dynamic formula as advanced. For piles driven with a Vulcan #1 hammer this formula is:

\[ B = 48 \log \frac{10}{s} \]  \hspace{1cm} (37)

in which \( B \) is the ultimate bearing capacity in tons and \( s \) is the customary set per blow for the last 6 in. or twenty blows, in inches. A safety factor of from two to three is recommended.

The application of statistical practice to pile driving has been inexcessably neglected. A concerted effort in this direction can yield a reliable pile bearing formula, based on the wealth of data already on hand, within six months.

The use of the wave equation to solve for driving stresses most certainly deserves serious consideration. As a result of Mr. Smith's intimate acquaintance with this subject he has devised a most ingenious analogy; the weights and springs concept. This expedient will simplify for many, who might otherwise have remained unaware, the principal of the wave equation. The use of electronic computers to facilitate the mathematical work is a natural consequence; and indeed is virtually a necessity. It is regrettable, however, that Mr. Smith has not yet compared his theoretical results with those obtained by other investigators from field tests. This suggestion was made previously\(^{29}\) and the source of valuable field data,\(^{30}\) obtained by affixing strain gauges to driven piles was cited. One of the many interesting results reported in the last cited reference is the small increase in stress at the head and sharp decrease in stress at the tip, with increased pile penetration and driving resistance. This should serve a signal of caution to those who would apply the wave equation without further investigation.

The application of a fundamental mathematical theory to practice is frequently negated because of the necessity to make at least partially invalid assumptions and over-simplifications. This bane is evident to some degree in Mr. Smith's paper. According to Cummings,\(^{19}\) the application of the wave equation to pile driving assumes that:

1. The sides of the pile are free and that there is no side friction which would affect the stress waves running up and down the pile.
2. Stress waves in the hammer may be neglected.
3. There are no flexural vibrations of the pile.
4. The pile behaves as a linearly elastic rod.
5. The hammer strikes directly on the head of the pile and that the surfaces of contact are two ideally smooth parallel planes.
6. The lower end of the pile is fixed.

In addition to these assumptions, the theory does not include the effect of dissipation of energy due to propagation losses in the pile.

Cummings' further comments should be studied before any attempt is made to apply the wave equation; albeit in his paper, too, certain concepts are not clearly explained. One generalization, based on Cummings' analysis bears repeating here. The six foregoing assumptions all lead to conservative answers. That is, the computed driving stresses are greater than actually occur. This fact has been substantiated by investigators in the field. Therefore, any attempt to correlate these computed stresses to bearing capacity will, naturally, yield unsafe results.

Looking now to the problem solved by Mr. Smith, we find that for the conditions specified, a Vulcan #1 hammer driving a 12BP53 to a count of five blows to the inch, at a pile penetration of 100 ft, develops a ground resistance to driving, at the tip, of 200,000 lb. Substituting the set, 0.2 in., in Eq. 37 yields a value of 164,000 lb. Considering that Mr. Smith's answer is too great for the reasons outlined previously, the likelihood is that Eq. 37 is as dependable as Mr. Smith's method. The compressive forces are shown as increasing rapidly towards the tip of the pile. This conclusion is diametrically opposed to the results obtained from the field tests previously cited.\(^{30}\) This same paper also concludes that:

1. The measured driving stresses in the top of the steel piles increased with an increase in pile penetration, and attained a maximum value equal to 20 to 34 times the static weight of the hammer ram with a 3-ft stroke.
2. Only about 7% of the driving stress measured in the top of the single friction steel pile was observed at the point of the pile.
3. Only about a third of the total driving stress measured in the top of the 110-ft steel pile driven through 90 feet of clay, silt and silt clay into sand was observed at the pile point.

If these conclusions and observations are correct, then the maximum stress which will obtain with a Vulcan #1 hammer is between 100,000 and 170,000 lb. The higher value is about 60% of the head value and 40% of the point value reported by Mr. Smith. The other two conclusions have already been discussed.

There are other questionable aspects in Mr. Smith’s method; most notable being the need to preassign not only the factor \( R_0 \) but also the penetration at which this value occurs. The ability to accurately predict what Mr. Smith implicitly takes for granted, would, in itself, be a major breakthrough in the field of pile driving analysis.

The ultimate solution to this problem lies in the field, the laboratory of the foundation engineer. No theory, however rigorous mathematically, can satisfactorily explain the pile driving phenomenon, unless it is modified to reflect the heterogenous nature of each pile driving operation.

Mr. Smith should enlist the aid of others, if necessary, to verify experimentally his theoretical conclusions and modify them accordingly. The fact that he has been slow to do so detracts only from the immediate application of his efforts. He must still be commended for laying an important building block in the theoretical area of foundation engineering.

E. Jonas,\(^{31}\) M. ASCE.—The author has rendered an important service to the civil engineering profession by providing a solution to the wave equation applicable to pile driving operations. His earlier paper reached only a limited number of civil engineers and, therefore, did not arouse the interest and discussion that it deserved. By applying the theory of longitudinal stress transmission to pile driving problems, the author has introduced a rational approach to the analysis of pile driving action under a specified set of conditions.

For determining the stresses in the pile at any instant during impact, the author uses a series of simultaneous equations, that can be solved by an elec-

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\(^{29}\) Letter to Mr. Smith, May, 1958.
\(^{30}\) Proceedings, American Railway Engineering Association, September, 1950.
A. A. EREMILN, 33 M. ASCE.—The author stated that by using the conception of the wave equation and resorting to numerical integration on electronic computer, a solution of the pile driving problem can be obtained that produces accuracy with about 5%. Further, in his conclusion Mr. Smith stated that on account of various limitations in the field of soil mechanics and construction materials, an accuracy of foundation design within 5% is prevented.

In design of foundation precision of 5% is seldom considered as criterion. Considering of vibrating forces in pile driving, nevertheless, has important value even at the smaller precision.

In the numerical illustrative example, Mr. Smith considered velocity of stress distribution in the pile material same as velocity of sound distribution in similar solid material. Pile is, generally, driven in the conglomerated structure of soil material. Therefore, it should be expected that velocity of stress distribution will be retarded by various disturbances due to changes in physical properties of soil and pile.

The author considered that the constants of damping forces varied from 0.05 to 0.20. The similar damping constants are, generally, assumed in the vibrated single units. In pile driving, however, the damping forces may be increased by considering combined effect of tip bearing stresses, buckling stresses in pile, and effect of elastic properties of soil.

Mr. Smith stated that numerical numerical data was obtained from driving of piles by vibrating. Numerical information on driving long piles by vibrating would be highly appreciated.

G. M. CORNFIELD. 34—The many existing dynamic pile driving formulas are admittedly not very satisfactory as has been shown by several investigators. It is, however, suspected that the correlation between the formulas and test data would have been improved if the investigators had eliminated certain cases to which it is known that dynamic formulas are not applicable, for example, pure friction piles in soft clays, and cases in which a re-drive test shows a larger set on redriving. Test loading is undoubtedly the best method of determining the ultimate resistance of a pile, but normally only a small percentage of piles on a site can be tested and there remains the problem of dealing with the remaining untested piles. Are these other piles to be driven to the same level as the tested pile, or to the same set, or to the same penetration into the main bearing stratum? If the problem is to be solved using only borehole and soil test data as a basis, it has to be remembered that soils can be very variable in their characteristics even over short distances and, to make matters worse, cohesionless soils such as sands and gravels will have their properties altered by the driving of every pile on the site. Thus, the test loading of a few piles is but a partial solution and there is, therefore, a real additional need for some reasonably accurate method of making use of the driving data, that is the hammer energy, set, and so forth, to determine the ultimate resistance of a pile in appropriate cases. It is to be hoped that the application of computers to the solution of the wave equation will fulfill this requirement as well as the case of small contracts for which testing loading may be an uneconomical proposition.

It would appear that there are two possible ways of applying the results of computer analyses of the wave equation. The first and obvious one is for the determination of the ultimate driving resistance of piles. However, it may take a relatively long time before sufficient comparisons of wave equation computations, test load data, and soil information will have been made to confirm that the method gives correct results, that the right constants are being used, and so on. A second type of application of the wave equation would be to use it to study possible trends in the pile driving process, for example, the significance of each of the parameters and whether the value of the ultimate resistance is sensitive or not to particular ones. This can possibly be studied without much further delay on the basis of such wave equation computations that have already been carried out. Mr. Smith, has shown examples of possible trends of the nature just mentioned.

Wave equation computations suggest that for steel bearing piles of normal weights the effect of variation in pile length may be insignificant in which the lengths are over about 50 ft. Thus, two steel piles of the same weight per foot, driven with the same hammer, and to the same set per blow may have approximately the same ultimate resistance, even though the driven lengths of the piles are 50 ft and 100 ft respectively. This has led the writer to compare the failure test loads of a number of long piles with the computed driving resistances obtained by the Hiley pile driving formula as he has not had the opportunity to use a computer. Eleven concrete piles (lengths 65 ft to 100 ft) and twenty-six steel piles (lengths 80 ft to 190 ft) were examined, and it was found that improved correlation as between the Hiley formula and the failure test loads resulted if, instead of using the actual pile length, an assumed artificial length of 50 ft was used in applying the formula for each case.

A diagram illustrating the application of the wave equation to steel bearing piles of various weights per foot, but of the same length and driven with the same hammer, to the same ultimate resistance is presented elsewhere. A significant trend that is apparent in this case is that over a wide range of pile weights, that is from about 60 lb to 300 lb per ft of length, the set required lies within the narrow range of 5 of 7 blows per in. This indicates that the pile weight is a relatively unimportant factor, for this particular example. It is interesting to note that the preceding examples of trends arising from the application of the wave equation suggest the possibility that the ultimate driving resistance of piles may not be very sensitive to variation of the weight per foot run of the piles and of the pile lengths for a given hammer drop (fall) and set. The most significant factors apparently influencing the ultimate resistance would then be the hammer ram weight and equivalent drop (fall) as well as the set. It is the writers' hope to examine this in more detail by making use of the results of a large number of failure test loadings of concrete and steel piles.

The writer looks forward to hearing more about the wave equation and in particular to seeing published data, possibly in the form of tables as mentioned by Mr. Smith in another publication.


Robert D. Chellis, F. Asce, and Arthur F. Zaskey—The valuable paper by Mr. Smith on a mathematical solution of the longitudinal wave equation for determination of pile driving resistances represents a great deal of painstaking study with consideration of the many variables, their effects, and methods of handling them. Availability of such a method of determining driving resistances may enable use of its results as a yardstick against which results by presently used formulas may be compared, as the author suggests. Load test results have been used as yardsticks, but they often are not available until piles or driving rigs are on the site for the start of driving, particularly for medium and small sized projects. Moreover, a load test result applies only to a particular set of conditions. Soil mechanics has provided methods for determination of soil capacities to carry loads applied by piles, such as (a) the sum of bearing and friction values determined by judgment, soils laboratory tests, or field tests of soils in place, and (b) total static soil load carrying capacities by the cylindrical pier method. Use of the wave equation may make possible a more accurate determination of probable driving resistance at an early date.

The author wisely limited the scope of his paper at this time to presentation of the wave equation solution. This discussion Comments on this method and also considers its possible use in practice. Relationships between the proposed method and present dynamic formulas and load tests are considered to aid in determining degrees of accuracy obtainable and expectable.

Computer programs such as the one described by Mr. Smith produce results that can not, as a practical matter, be attempted by hand. Therefore, disclosure of the detailed meaning of such procedures in a field of engineering must be relegated to those who possess the program. Development of such a program will cost several thousand dollars. Before making this investment, an engineer will want to compare results of finite difference computations to those of conventional methods, such as the Engineering News and Hiley formulas.

The familiar "engineering formula" must, for brevity, be summary or synoptic. It is contrived by artful omission. (Sequence of more and more omissions and the meanings and effects appear elsewhere, The Hiley formula is numerically frugal, refers where possible to observables, such as permanent set, and rebound, and accepts the discipline of budgeting the energy of hammer. The Engineering News formula is even more frugal of terms and is too simplified for the broad ranges of modern hammers and piles; for instance, it has no terms to express such wide variables as pile length or weight.

By contrast, the finite difference procedure develops a detailed impulse-momentum model. It is not summary or synoptic; it mimics the motion and stress of the piles.

For one process of computation to produce better results than another, it must utilize more information, or better information, or else make better use of the information that it has. A synoptic formula may merely omit from consideration information that is not available anyway. Opposing this is the possibility that engineers have failed to develop information for the reason that common methods are not set up to make use of it.

Comprehension of the principles of soil mechanics is needed when considering applicability of the wave equation, as well as of any pile-driving resistance formula or other method for obtaining load carrying capacity. Use and interpretation of such methods still remain partly science and partly an art. This is one of the principal reasons why the writers have advocated the use of the Hiley formula, backed by static investigations of end bearing and side friction. These methods require thought and consideration of the variables affecting driving and static bearing capacity and develop judgment or a "feel" for the relative effects.

The author mentions that, after a pile has been driven, the soil may largely retain its original supporting value or it may "set up" or "relax" around a pile. Also, effects of negative friction from gradual pickup of additional load on a pile from firm upper strata overlying incompletely consolidated soft strata may occur and continue with time. This action is not evaluated by dynamic or wave equations, but must be considered from the standpoint of soil mechanics, because long periods of time are involved.

The author defines over thirty terms, many of which are variables, whereas others vary only from case to case. Such a term as "J," the damping coefficient, can also vary with soil stratification and effects of pile taper. The damping resistances are only in action during driving and do not carry permanent load. Some uncertainty surrounds the value of "E" in reinforced concrete piles, as it varies with mix, degree of curing, and age. Electronic computers have the ability to handle the mass of cases resulting from so many terms. Determination of a method of presenting this information requires study.

Comments on Author's Conclusions.—Conclusion 1 - The author states that the proposed method gives permanent set per blow, as well as instantaneous stresses for any specified conditions. The terminology of "permanent set" may be ambiguous and might mislead if the use of the work "permanent" is referred to mean that the method gives the long-term static load that can be carried without further settlement. The intent of the conclusion is assumed to mean that "permanent set" is the net penetration per blow (plastic displacement, that is the irreversible portion) remaining after elastic rebound from that blow has occurred.

Conclusion 2 - The author indicates that the knowledge of soil mechanics under pile driving is incomplete. No matter how mathematically accurate driving resistance computations may be, they are no better than the assumptions used. Soil conditions at various borings may differ and between borings may differ from those at borings.

Conclusion 3 - The use of Micarta cap blocks is a valuable development. It is understood that a large number of piles can be driven with one block. This avoids considerable rapid changes in energy losses and sets that occur when the frequent replacements of wood blocks are necessary. A method for disposing of a variation that has generally been ignored and that affected the resistance of a pile during its driving and also the uniformity of comparative results between various piles. More information on this subject would be useful because the use of such cap blocks would seem to be advantageous to the engineer, pile driving contractor, and owner.

Conclusion 4 - The author states that the method may be used to determine the ranges of application through which other formulas may be considered reasonably accurate. To serve as a calibration standard in this way, a body of comparisons of wave equation results with load test capacities should be avail-
those at the close of initial driving. A spread of agreement should, as a practical matter be considered as normal and unavoidable.

If it can be determined that the Hiley or EN formula results are safe and that ultimate driving resistances are in reasonable agreement with wave equation results in any general range of conditions, then such formulas might per-

missibly be used within such limits. This might enable such simple formulas to be quickly applied in the office or field, thus avoiding the necessity of access to a computer in order to be sure of obtaining sufficiently reliable and economic results. Furthermore, establishment of a computer program for solving resistances by the Hiley equation is quite simple, provided that the volume of work warrants it.

The writers have computed the driving resistances of 80 load-tested piles by the Hiley formula. Effective lengths were full lengths for end-bearing piles and assumed lengths to centers of driving resistances for piles carried partially or entirely by friction. The permanent load-carrying strata in all cases were sand or sand and clay. Test load values have been plotted against driving resistances in Figs. 13 and 14. If these values are identical, the points of intersection should fall on a 45° line. Lines representing a 25% variation each way from the line of coincidence form an envelope that contains most cases. This should be satisfactory when using a factor of safety such as 2.5 and would result in a range of 1.9 to 3.1, or say 2 to 3.
FIG. 15.—WAVE EQUATION AND ENG. NEWS AND HILEY FORMULAS 14 INCH OCTAGONAL PRECAST PILES—NO. 6 VULCAN (SAME FOR 60C) END BEARING

FIG. 16.—WAVE EQUATION AND ENG. NEWS AND HILEY FORMULAS UNIFORM STEEL PILES—NO. 1 VULCAN (SAME FOR 50C) END BEARING
Agreement between driving resistances computed by the Illey formula and load tests appears to be satisfactory for the piles shown in Figs. 13 and 14. In a few cases, load-test results were appreciably greater than the Illey formula results; such cases seem to have been characterized by piles that were both long and heavy, although computed values of some piles both fairly long and heavy were in good agreement with load test values. Computed driving resistances of piles, either long or heavy, but not both, showed good agreement between computed driving resistances and load-test values. The number of cases shown is not enough to state that the above effects will always hold true, but the results are suggestive of possible relationships.

The Illey formula and test load results, compared in Figs. 13 and 14, show groupings fairly well averaged above and below the line of exact agreement. Wave equation curves such as shown in Figs. 15 and 16 appear to lie in somewhat higher ranges of driving resistance values than do the Illey curves. These curves are all for full end-bearing piles. Relatively small driving resistance differences appear between short and long piles and occur when using the wave equation as indicated in Figs. 15 and 16, whereas driving resistances decrease quite materially for longer piles when using the Illey formula. This tends to indicate that the ranges of wave equation values might be too high, and that lower results would agree better with load tests.

It should be carefully noted, however, that the wave equation curves shown in Figs. 15 and 16 represent the result of calculations made 4 or 5 yr ago before the damping constants J and J' were introduced. Use of these damping constants, as recommended by the author, would give lower values and thus bring the wave equation results into closer agreement with the Illey formula except probably for piles that are both long and comparatively heavy. At the present time an electronic computer at the Agricultural and Mechanical College of Texas is being programmed for pile calculations by the wave equation, and in due course the results of these calculations will be made available. At that time more accurate comparisons can be made.

The author mentions another publication as indicating that the Newtonian theory of impact does not apply directly to pile driving. The same publication also states that the Illey formula includes some losses twice; however, the wave equation and Illey curves shown in Figs. 15 and 16 show remarkable similarities in shape. They apparently belong to the same family. If the wave equation results are proved to be correct, it might be that proper coefficients inserted in the Illey formula would give close agreements, thus giving a simple solution that can be applied to any case without use of the wave equation.

Consideration of methods for obtaining wave theory solutions and of presenting results in the most economical and convenient manner will be needed. A very large number of graphs would be required to cover the ranges of variables used in the wave equation. Fig. 15 shows a set of curves for driving resistance versus blows per inch for three lengths of the same uniform cross section of end-bearing precast concrete piles driven with one type and speed of hammer; such a graph applies only to these conditions. These curves, furthermore, are based on using a constant selected value of hammer efficiency, pile modulus of elasticity, cap block coefficient of restitution, and elastic constant. A separate set of graphs or tables would be required for each combination of driving hammers and pile types, weights, and lengths. Step-taper pile cores are available in many combinations of diameters and weights that would require a further multiplicity of tables or graphs. Precast concrete piles are used in many shapes and sizes, solid or hollow with various moduli of elasticity and amounts of reinforcing steel. Followers and composite piles, if used, require special consideration. If all or part of the driving resistance is to be met in side friction, further sets of tables and graphs will be needed. If the designer wishes to obtain values other than can be found from such graphs or tables, if these become available, it would be necessary to make several interpolations or establish a program for an electronic digital computer. Some engineering organizations might own or rent a computer, but it is more likely that the problem would be taken to some computer computing service to perform at their standard rates.

Tables of pile resistances computed by the Illey formula are contained in a book by R. V. Allin. These tables are based on driving square wood or concrete piles with British hammers only and are types not used in American practice. They consider relatively few variables compared with the number involved in the wave equation but are mentioned as indicating the difficulty of presenting wave equation results in such a manner.

A nomogram based on the Illey formula has been prepared by G. M. Cornfield. Six variables are used. This indicates the difficulty of preparing or using nomograms containing the large number of variables involved in the wave equation.

A possibility of interest is a computer program based on statistical correlation between the number of blows per foot in driving a sampling spoon together with the other information known in advance of driving and with load tests to failure known only after driving.

Future Procedures.—It seems desirable to explore the place of the proposed method in the entire problem of foundation design.

The proposed theory of longitudinal impact gives only driving resistances of individual piles. It does not consider the value of the supporting soils or the effects of group action any more than do any dynamic driving formulas. This factor alone may serve to make considerable changes in the allowable pile load, but the various methods of computing reductions show a wide range, indicating so large an unknown that great accuracy in predicting pile carrying capacities may not be possible.

In future years, many engineers may become familiar with electronic digital computer programming and use such a method extensively, but for some years at least many pile problems will continue to occur in offices or in the field in which computers or programmers may not be available. Unless, or until, universal acceptance of this method takes place and means for its use are available and practicable for each combination of factors, the degree of reliability of the current methods for all types and sizes of piles and hammers remains of interest.

It is hoped that the subjects of correlation of the wave equation results with test load values and with the Illey and Engineering News formulas will be carried forward and results publicized, including those for full or partial piles.

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Soderberg's discussion suggests certain changes in the method of computation that would not materially affect the final result but are intended to reduce the amount of computation required. It is characteristic of approximate solutions that there is usually a choice as to how the approximation is to be made; Soderberg has developed a method somewhat different from that of the writer, and has set forth its essentials in his analysis. He should have full credit for this development.

Soderberg computes only the displacements. Forces or other information that may be wanted must be obtained by means of a supplementary computation, therefore in many cases the time saving might be much less than Soderberg has indicated. In any event, this is a subject for study primarily by mathematicians and computer experts as it does not affect the civil engineering aspects of the problem.

Gates examines the question of how the bearing capacity of piles should be determined. For this purpose Gates opposes the use of the wave equation, but advocates a pile driving formula (Eq. 37) which he has devised. The writer presented this general subject briefly in the paper under the headings "Introduction" and "Recommendations and Comments." The words were carefully chosen, and there is no wish to add to them herein or to amend them in any way.

In Gates's discussion there are three statements that call for answers as follows:

A. The discussion states that previous investigators "unanimously cautioned against the use of the wave equation as a means of determining bearing capacity." No specific references are given to support this statement, and the writer has been unsuccessful in his attempt to confirm it. He did, however, find a statement exactly to the contrary in the writings of A. E. Cummings from the 1930s. This statement appears in the last part of a paragraph on page 13 of Gates's reference No. 25. Cummings first refers to the wave equation developments of St. Venant, Boussinesq, Isaacs, and the British Building Research Board (Gates' references Numbers 11, 12, 17, and 18) and then writes as follows: "However, there is a considerable amount of field evidence available which shows that the stresses transmission characteristics of a pile are of great importance not only in determining its behavior during driving but also with respect to its subsequent ability to carry static load." Cummings' paragraph concludes: "It is a new and promising field for investigation."

B. It is suggested in the discussion that the writer "should make a definitive statement concerning the limits of application of his presentation." The writer has no intention of doing this because he believes that there is no good reason for limitations beyond those already stated in the paper.

C. It is stated in the discussion that the writer is guilty of "qualification of his previous references to bearing capacity by mentioning in this paper soils which are relaxed or set-up." However, the writer had previously mentioned these particular soil characteristics in the second paragraph of Gates's reference No. 18 published in 1937. No such qualification has occurred.

Gates has implied that the wave equation and a dynamic pile formula are the same, and that references whose authors had no thought of the wave equation may therefore be cited against it. The wave equation is a mathematical method. Dynamic pile formulas appear to be mathematical, but their mathematics is defective as has been carefully explained by Cummings in two separate articles devoted to the subject.

Passing completely from the subject of bearing capacity, the analysis next proceeds to criticize four technical features of the paper. In all four cases the criticism makes statements which, if taken together, leave the reader with the impression that the writer's method is quite inaccurate. These statements are presented in the following:

No. 1: The discussion lists six assumptions necessary in order to apply the wave equation to pile driving. These have been taken from a list18 that applied specifically to the wave equation pioneering done almost 100 yr ago by St. Venant11 and Boussinesq.12 This list is now (1961) out-of-date. Only assumptions Nos. 3 and No. 5 are necessary in accordance with the writer's method. Assumption No. 1 can be made if it appears to fit the actual ground conditions, but it is not necessary because side friction has been specifically provided in the writer's method. Assumption No. 2 involves negligible error if the ram is short and stubby, but a comparatively long ram may be represented by any suitable number of weight and springs as per Fig. 10(b). Assumption No. 4 is not necessary because piles of irregular section have been provided for. Assumption No. 6 is completely unnecessary.

No. 2: Immediately following the list of assumptions it is stated that the writer's method takes no account of propagation losses in the pile. The writer has presented Eq. 27 for this purpose. However, in piles of ordinary length the internal losses are negligible.

No. 3: Near the end of the discussion, the compressive forces computed in the writer's Illustrative Example are compared unfavorably with those obtained by strain gage measurement in certain field tests. The discussion emphasizes that the computed forces increased rapidly towards the tip of the pile, and states "This conclusion is diametrically opposed to the results obtained from field tests." Gates has ignored the fact that in the Illustrative Example, side friction was completely absent, whereas the field measurements were made on piles in which side friction predominated. There is no reason whatever to expect the forces to be distributed in the same manner in these two radically different cases. The discussion also emphasizes that the computed forces are larger than the measured forces. Here again a vital fact is ignored, namely that in the two cases the pile weight per foot is different. The heavier and stiffer pile used in the Illustrative Example will inevitably produce higher impact forces if the same hammer is used. A comparison of this kind should not have been made because the only major way in which the two cases resemble each other is that the same hammer was used.

No. 4: The discussion states that in the writer's method there is a "need to preassign not only the fact that the penetration occurs." As a matter of fact the penetration "s" is obtained by making the wave equation computation. It is never under any circumstances "preassigned." This appears in the paper as part of the first paragraph under the heading "Outline of Numerical Method" and also as part of "Conclusion No. 1." It also appears in "Step No. 9" of the Illustrative Example where it is stated that the computed permanent set "s" is 0.20311 in. per blow.

Jonas has written a thoughtful discussion which raises an important question, namely: should $R_u$ be 200,000 lb as given in the paper, or should $R_u$ be 400,000 lb or 400,000 lb as suggested in the discussion.

The discussion states: "It can be shown that in the Illustrative Example given by the author, equilibrium does not obtain at the end of permanent set unless Eqs. 13 and 16 are modified."

In the Illustrative Example the "end of permanent set" occurs approximately in time interval 55 when the point of the pile reaches its maximum penetration, as can readily be seen by referring to Fig. 12. At this instant $F_{11}$ (the internal force at the point of the pile) does not have its maximum value of 405,000, as quite reasonably assumed by Jonas, but has a value of only 200,000 lb which is the same as the value of $R_u$ (the ultimate ground resistance). This fact can be determined by reading values from the curves of Fig. 12. Thus in time interval 55 $D_{12} = 0.202$ in., $D_{11} = 0.302$ in. and $D_{11} = 0.353$ in. Subtracting $D_2$ from $D_{12}$ gives 0.1 in. as the measure of the elastic ground compression. Multiplying 0.1 in. by the spring constant of the ground $K_2 = 2,000,000$ gives the value of $R_u$ as 200,000 lb. Subtracting $D_2$ from $D_{11}$ gives $C_{12}$ as 0.051 in. Multiplying $C_{12}$ by the spring constant of the point of the pile $K_{11} = 3,900,000$ gives $F_{11}$ as 200,000 lb also. Thus equilibrium is obtained.

The maximum value of $F_{11}$ (405,000 lb) occurs not at the end of permanent set but in time interval 34 when the point of the pile is forcing its way through the ground at the rather high velocity of 6.4 fps. An analysis similar to that given in the preceding paragraph but involving resort to the printed computer output, shows that in time interval 34 the following values occur:

- Ultimate ground resistance $R_u$ ...... 200,000 lb
- Viscous ground resistance ........ 182,000 lb
- Accelerating force acting on $W_{12}$ ........ 12,000 lb
- Total ........ 405,000 lb

This total is exactly equal to the maximum internal force in the pile, $F_{11}$.

Therefore, it may be concluded that the correct value of $R_u$ is 200,000 lb and that no modification of Eq. 13 and 16 is required.

Eremin's discussion raises a question regarding the velocity of sound (or stress) in a pile. None of the equations used in the writer's method includes, as a term, the velocity of sound. Nevertheless, as the computation is performed, the stress wave progresses down the pile with approximately the velocity of sound, as it should. Furthermore, if side resistance is included, the stress wave will be slowed down automatically as Eremin states that it should be. This is an important proof of the correctness of the method.

The values of 0.05 and 0.15 proposed for the damping constant $J'$ and $J$ are the best now (1961) available. When experimentation or other means produces more accurate values, they should be adopted.

Eremin suggests that "tip bearing stresses," "buckling stresses," and "elastic properties of soil" have an effect on the damping forces. The effects of the first and the last of these are considered in the paper and result in Eqs. 15 and 16, and Eqs. 28 and 29. After the pile enters the ground there is little tendency to buckle if the pile is reasonably straight, therefore the effect of buckling stresses has not been considered in the writer's paper.

Eremin refers to data obtained from driving piles by vibration. No mention was made of such data in the paper.