A FULL SCALE INSTRUMENTED PILE TEST IN THE NORTH SEA


ABSTRACT

During the early Spring 1976, BP carried out tests on an instrumented working pile on their FD structure of the Forties Field (British Sector, North Sea) in 121m (400 ft) water depth.

The test was carried out on a 100m (330ft.) long pile 1.37m (54 in.) diameter. The purpose was to provide data on the influence of the many constraints on the transmission of a pile driving stress wave with a view to eventual improvement in analytical tools using wave equation techniques.

The test pile was instrumented with strain gauges, accelerometers and pore water and total pressure cells at various levels.

Successful connection was made with the instruments after pile stabbing and was maintained for the 14 day duration of the test, a period during which severe storms were experienced. The data has been interpreted in engineering terms for selected blows of different hammers and compared with theoretical predictions using both conventional and modified wave equation analyses.

Good correlation was obtained between recorded and predicted values for the acceleration measurements and reasonable correlation was achieved for strain measurements during the early stages of driving. Total and pore water pressure measurements were made at three levels on the pile and the results are presented in the paper.

INTRODUCTION

Steel tower or template type platforms for offshore drilling and production are usually supported on piles of the fully driven type. The assessment of their bearing capacity and the prediction of their drivability has entailed considerable extrapolation of techniques, often based empirically on data recorded during onshore driving. Useful predictions of drivability can be made by means of the wave equation analyses but the influence of sleeves, guides and connectors cannot be adequately modelled until their behaviour is better understood. The planning of the installation of an offshore platform entails placing great reliance on such predictions at a very early stage.

In an effort to provide a basis for improved predictions B.P. Trading Ltd (BP) decided to attempt measurements on a pile during its driving in the Forties Field. Early in the development of this Field, BP had retained Texas A and M University, to measure the stress level during driving within the chaser immediately below the hammer. Such measurements can now be reliably performed with a minimum of interference to the installation contractor and can provide a rapid 'on site' check of hammer performance, using either a hard wire or telemetry system. Measurements of the behaviour of the pile itself and, in particular, those relating to its embedded portion are considerably more difficult to achieve than measurements made on the chaser above water level.

The Forties Field is in Block 21/10 of the British Sector of the North Sea where the water depths range from 107m to 128m (350 to 420 ft). The soils in the region of the four steel platforms consist mainly of silty, lightly over-consolidated clays with layers of dense sand. At some locations these sand layers have, by their thickness and elevation, greatly influenced pile drivability.

The Forties jackets are sensibly square in plan, the piles being clustered around the four corner legs. Sleeves and guides are available at each leg for accommodating 12 piles on the FA and FC platforms and 10 piles on platforms PB and PD. Prefabricated single length open ended pipe piles up to 100m long (330 ft) were employed on these jackets, driven to maximum penetration depths of up to 76m (250 ft). All piles were of 1.37m (54 in.)

References and Illustrations at end of paper.
outside diameter and 50mm (2 in.) wall thickness with a thicker walled portion (2 in. or 62mm) over the length required to withstand the maximum bending and shear stresses just below mudline.

Chasers (or followers) were used during pile driving made up to the required lengths with Rockwell connectors and located on the pile using gravity (i.e., free) connectors. At the completion of driving all the piles or, in some instances a group of piles, were grouted into the sleeves. Fig. 1 presents an elevation of a jacket showing the location of the pile guides and sleeves and indicating typical chaser lengths used during driving.

BACKGROUND TO TEST

During the initial planning of the piling at Forties, the possibility of performing measurements was considered since results from pile driving studies performed at Nigg Bay (Ref. 1), and during installation of the West Sole Platform (B.P. Gas Field, Southern North Sea) were being extrapolated to aid predictions for the Forties Field Platforms.

Early in January 1976 BP decided to continue the study by instrumenting a single pile on the FD platform (the last of the four in the Forties Field to be installed). This pile would be extra to the foundation requirements of the structure and would have a total length of 100m (330 ft).

The soil strata below the platform consists of normally consolidated or lightly over-consolidated silty clay without significant sand layers over the upper 52m (170 ft) underlain by more highly over-consolidated clay to the full piling depth (250 ft or 75m). Consequently, pile driving to full penetration depth could be achieved without much difficulty. Borehole logs for the relevant borings are shown on Fig. 2 together with the measured soil strength profile. Further information on these aspects are presented in Reference 2.

The Forties FD jacket was constructed by Highland Fabricators at their Nigg Bay site. Brown and Root Inc., were the installation contractors and their derrick barge Hercules was used for all the foundation work on the jacket.

The type and location of the instruments are shown on Fig. 3. They include four discrete types of measurements; axial strain, axial acceleration, total pressure and pore water pressure. A high speed camera was used to prepare photographic documentation of the behaviour of the hammers from which hammer impact velocity and short strokes could be assessed. The final extent of the study was governed by the number of individual conductors available in the main cable employed. Whilst measurements at pile extremities were included in order to maximise any differences that may result, most of the sensors were placed over the lower 48.5m (160 ft) to examine geotechnical rather than structural influences on pile behaviour.

TEST PROCEDURE

Making measurements on the embedded portion of an offshore pile during its installation by driving in 140m (460 ft) of water presented a considerable challenge. Pile installation procedures had to be developed which permitted an electrical connection to be made to the pile with a minimum of delays to the overall operation. The detailed planning of the study was, to a great extent, governed by the constraints set by this procedure.

At that time no adequate means were known whereby electrical connections could be made underwater without generating sufficient interference to completely obliterate the low level signals involved. Consequently, the signal cable had either to be paid out during pile stabbing and subsequently threaded through chasers, or stored within the pile for retrieval immediately prior to driving. The vulnerability of the cable during the former procedure led to the adoption of the latter.

In outline, this method involved linking all the transducer circuits to the main signal cable at a junction box within the pile and storing the remainder of the signal cable within the upper portion. On completion of the pile stabbing operation and of the placing of the main chaser this signal cable could be drawn through the pile train and out through the prepared hole in the chaser side to link with the recording equipment on the derrick barge.

Instrument Installation

The layout of the various instruments within the pile is shown on Fig. 3. These included:

(a) Accelerometers (P.C.B. model no. 305A 04). The use of accelerometers in this study was based on the proven reliability of this type of sensor. The low mass piezo-electric type with pre-amplification was selected. Accelerometers were mounted in pairs on small steel brackets such that their axes were aligned parallel to the pile axis and at a distance not exceeding about 12m (0.5 in.) from the pile inner face. These brackets were designed in such a manner that their own resonant frequency would not interfere with the experimental data.

(b) Strain Gauges (Alitech Model NG129)

The weldable type of strain gauge was selected in preference to the foil type for this study because of its greater degree of tolerance of adverse ambient conditions during installation. Such gauges have been used for high shock measurements of short duration without apparent loss of accuracy. Each strain gauge bridge consisted of three gauges mounted axially at 120° intervals around the circumference of the inner wall of the pile at each measuring level. These were series connected to form a single arm of the bridge and were linked to the bridge completion unit at the monitoring station by a three wire arrangement.
(c) Total and Pore Water Pressure Cells

These cells were of the strain gauge diaphragm type. The sensing surfaces of the total pressure cells were nominally flush with the outer surface of the pile.

The face of the pore pressure cell was covered by a flush mounted sintered stainless steel disc. The cells were fixed into rings which were welded within holes in the pile wall. The cells were installed at three levels in pairs, each pair consisting of a total pressure cell and a pore pressure cell mounted diametrically opposite.

A standard 100m (330 ft) pile was modified onshore to receive the instruments. Conduits, formed from angle section steel, were continuously welded to the inner pile wall to provide cable protection. In order to minimise the probable influence on plug behaviour of inner obstructions, no circumferential conduits were installed within the lower 30 metres (100ft) of the pile. Protective ploughs were welded on the inner wall to reduce scour of the rear of the pressure cells and to protect the cable running between each cell and the adjacent conduit. A steel base plate was fixed 82 metres (270 ft) above the pile tip for the main electrical junction box and a cable supporting cage was installed 2 metres above this consisting of a ring beam supporting an annulus of expanded metal.

Strain gauges and accelerometer were mounted on the pile wall within the conduits. All the in-pile wiring, consisting of four core heavily armoured type cable, were terminated at a junction box and connected to the main transmission cable. This specially designed junction box contained neoprene seals for all input and output cables and the connection regions were individually insulated and located freely within an oil filled chamber.

The 82 core main transmission cable (specially made by De Regt) consisted of two layers (38 and 44 core) of 0.37mm diameter stranded copper wires covered with polyamide and filled with a water proofing rubber compound. These conductors were arranged around a central steel core with a breaking strength of 100kN (10 tons) and were surrounded by an inner jacket of polyurethane, a steel screening double helix and an outer polyurethane jacket, the overall outside diameter being 29mm. The free end of the 228m (750 ft) length of this cable was terminated with an 85 way connector fitted with a water proof sealing cap.

At the completion of the wiring all conduits were injected with a neoprene compound to seal and secure the cables.

Monitoring Equipment

The monitoring equipment consisted of a signal conditioning unit accompanied by a signal storage system which was all pre-wired under laboratory conditions to an 85 way socket recessed in the unit's front panel. The signal storage equipment consisted of three multi-channel magnetic tape recorders and a high speed ultra-violet recorder.

All the pressure transducers were connected to a single seven channel recorder thus allowing a slower tape speed to be used since a high frequency response was not required for these sensors. Of the remaining channels, priority signals were linked to a fourteen track recorder and the remaining duplicate signals to another seven track instrument. A further portable data logger was constructed for automatically recording the pressure values during the set up period.

Offshore Handling Procedure

The pile was transported from the fabricating yard where the instruments had been installed and transferred at sea to the derrick barge Hercules. Final checks were made on all the instruments and cable stowing was then commenced. The 228m of signal cable was packed tightly within the upper region of the pile in a folded figure of eight manner which would permit twist free withdrawal at a later stage. A 20m length of 12mm wire rope was attached to the free end of the cable by means of a clamp incorporating protection to the 85 way plug. The free end of this messenger wire was spliced to a release unit attached to the inner pile by a bolt passing through from the outside. Finally a further 8m messenger rope was attached to the other end of this unit and was coiled immediately inside the upper end of the pile and held with hemp rope. All these precautions were necessary to avoid damage to the main cable during the removal of the 150mm main slings used to stab the pile. Pile stabbing was performed in the normal manner which entailed passing the pile through a number of guides under diver control. The standard 14m (298.5 ft) chaser had been adapted for this study by the cutting of a 50mm hole in the wall at a preset elevation, the insertion of a bell shaped guide immediately below the hole and by fitting of both internal and external pad eyes at the hole. A 130m messenger wire was attached to the external pad eye, and passed over a floating sheave into the chaser and down to the gravity connector where the excess was coiled and tied to lugs near the mouth of the connector with hemp rope.

The chaser was stabbed fully in the conventional manner to ensure that the diaphragm had been pierced and that full self weight penetration of the pile occurred prior to cable handling. It was then raised 10m (30ft) to allow a diver to withdraw the coils of messenger wire from the pile and the chaser and to connect them. A tugger winch that had previously been connected to the sheave was used to draw up the messenger wire until tension against the release device was confirmed by the diver. The chaser was re-stabbed whilst maintaining tension on this wire, and the diver then unbolted the release device which permitted the drawing out of the signal cable. Sufficient cable was drawn out to adequately reach the instrument house and a split clam was attached to it at the follower outlet such that the hanging weight of cable was subsequently suspended on shock absorbent supports.
DRIVING OF TEST PILE

File driving took place in three main drives followed by a redrive after a suitable interval. The full driving record is presented in Fig. 4.

The majority of the driving was performed using the Menck 3000 steam hammer.

At the completion of these three drives the main transmission cable was carried onto the jacket and lashed firmly to the pile guide. The battery operated magnetic tape cassette data logger was connected to the cable on this guide to monitor the total and pore pressure changes during the set up period.

After a period of 14 days the redrive test was performed. Despite very rough weather within this period the cable appeared intact above sea level. When contact was re-established between sensors and monitoring equipment it was found that some strain gauge circuits could not be balanced. The redrive was commenced using the same chaser configuration as for the last main drive.

Detailed driving data was recorded for the early stages of the redrive, the blow count being noted for every inch of penetration for the first foot. A visual assessment suggested that the efficiency of the Menck 7000 was low and erratic during this early period, although the hammer had been heated up on deck prior to the test. The test was terminated after 2.0m (6.5 ft) of additional penetration.

During the test records of each sensor were made and stored on high speed magnetic tapes using the three tape recorders. An estimated 800 of the total of 2264 blows applied were monitored including each of the 212 blows of the redrive test and about 60 groups of ten blows at regular intervals. Pressure transducers were monitored continuously. In addition independent unsynchronized measurements of stress levels within the upper chaser were made for drives 2 and 3. For the redrive these sensors were wired directly into the central monitoring station thus ensuring that data was fully synchronized.

A verbal commentary together with notes prepared by various observers independently assessing such factors as the blow count and the shortfall enabled detailed study of specific blows.

During the test progressive malfunction of some sensors was observed. Two types of circuit failure occurred, the first and most obvious being the complete loss of signal and the second being the reception of a meaningless signal. Fig. 5 shows the periods of penetration over which sensible signals were detected from the various circuits.

Sensor malfunctions during the main drives often arose from excessive drift causing readings to extend beyond the available range of adjustment. In certain cases modification of the circuit was possible prior to the redrive to enable such circuits to be balanced.

Four high speed (220 frames per second) films were taken at various stages during the test at known blow numbers.

ACCELERATION AND STRAIN RESULTS

Over ten thousand acceleration and strain records were obtained during the course of driving. Due to space limitations only one typical set of data is included here. It is qualitatively similar to that obtained for all recorded blows.

For purposes of establishing the validity of any theory of pile behaviour during driving, it is necessary to compare predicted and observed time based records of one of the following variables: displacement, velocity, acceleration and strain. In the tests considered here, two were observed so that the experimental data could be corroborated and, in the event of a partial instrument failure, the duplication of data would ensure that no vital information was lost.

To obtain displacement or velocity readings would require a stationary reference point to be established near the test pile. For work over water this is clearly difficult. The need for a fixed reference point can be dispensed with if acceleration is measured using a small inertial device fixed to the pile wall.

For the comparison of actual and predicted behaviour acceleration is the least favourable variable which can be used while displacement is the best. This is because acceleration is the second derivative of displacement with respect to time. Observed values of acceleration are likely to be more sensitive to experimental errors while theoretical values calculated by the wave equation are subject to the greatest numerical errors. The comparisons are thus being made under more stringent conditions than is the case when either velocity or displacement are used.

Referring to Fig. 6 this shows the observed acceleration time graphs for points at 96.6m (317 ft), 48.8m (160 ft) and 16.7m (55 ft) from the toe of the pile. The accelerometer at the 317 ft level is therefore closest to the hammer while that at the 55 ft level is furthest away. The figure shows clearly that instruments nearer to the hammer experience the disturbance before those further away. The delays in time show that the disturbance is propagated down the pile at the velocity of sound in steel.

Fig. 6 also shows that the accelerations at the 317 ft, 160 ft and 55 ft levels appear to repeat themselves after 38 ms, 19 ms and 7 ms respectively. These correspond to the times taken by the disturbance to travel to the toe of the pile and then back up again. This infers that the pile is subjected to a wave type of motion.
Fig 7 shows data obtained from strain gauges on the pile wall. Because of the higher noise levels in these circuits the propagation of the stress wave down the pile is not as apparent. However, the data does show that the stress levels diminish as the toe of the pile is approached. This would be expected for a friction pile.

The accelerometer and strain records show that, following impact from the hammer, a strain wave is propagated down the pile. It is therefore, necessary to use the wave equation method of analysis. This was first suggested by Smith (1960) (Ref 3). In this method the continuous distribution of pile and soil are represented by a finite set of elements. The forces on each element due to the elasticity of the pile and the soil interactions are obtained and the equation of motion of each element formulated. This gives a set of second order ordinary differential equations which, due to the soil behaviour, are nonlinear. This makes it necessary to use numerical methods for their solution. Computer routines for this purpose are well known and will not be considered further here. Although the procedures are simple to use and can in theory accommodate very sophisticated nonlinear representations of real behaviour, it has been found that introducing additional complexities such as discontinuities due to connectors may create problems associated with the stability of the numerical solutions generated. Details of the methods for incorporating the behaviour of gravity and Rockwell connectors are given in Appendix 1.

The static resistance of the soil \( r \) is assumed to obey an elastic-plastic law of the type shown in Fig. 8. A dynamic component of soil strength is incorporated into this as suggested by Smith (1960). (Ref 3). If \( v \) is the velocity of the pile wall relative to the soil, the dynamic resistance of the soil \( R \) is given by

\[
R = r (1 + J'v) \quad \ldots \ldots \ldots \ldots (1)
\]

The parameters \( R, Q \) and \( J' \) must be evaluated for conditions of deformation and failure corresponding to those of the actual pile. (where \( R \) is the ultimate static soil resistance, \( Q \) is the soil quake and \( J' \) is the damping constant). This required \( R, Q \) to be obtained from a test in which the instrument simulates the actual pile. Similarly \( J' \) must be obtained from model pile tests where the pile is driven at known velocities into the soil. A laboratory machine for this purpose has been described in Ref. 4. The values of \( J' \) and \( J \) (point damping) used in the theoretical predictions are 0.20 and 0.01 sec/ft respectively and \( Q \) was taken as 0.1 in.

The results of theoretical computations for blow number 122 are shown in Fig. 7 (b), 7 (c), 9 and 10. Figs. 7 (b) and 9 are conventional wave equation analyses in which the presence of connectors is ignored while the results shown in Fig. 7 (c) and 10 take connectors into account. Prior to the blow being struck all the connectors were assumed closed. The soil resistance at the toe of driving (SDR) was taken as triangular. Its magnitude was adjusted to give a set per blow matching the observed. For this particular blow the Menck 3000 was assumed to be operating at an efficiency of 47%.

From an examination of Figs. 6 to 10 and many other similar records not included herein, the following general comments can be made on the comparison between measured and predicted behaviour:

(a) amplitude variations of the prime pulse can be attributed to incorrect modelling of the soil resistance.

(b) the amplitude of the reflected waves of the recorded signals often exceed those of the prime pulse. This was not predicted in the conventional wave equation analysis but was often indicated in the modified version. It is therefore thought to be due to connectors opening and closing.

(c) as mentioned previously the propagation of the hammer blow down the pile is clearly defined.

(d) the high frequency component of the acceleration spectra predicted by the modified wave equation solution were absent in the measurements. This can be attributed to the limiting band width of the tape recorder (1.4 kHz). In each case the measured profile shows an averaged response to the high frequency component.

(e) electrical interference affected the 160ft level as can be seen in Fig. 6. No reason for this has been found.

(f) much less data was obtained from the strain gauges. This is because the signal to noise level ratio was low. If noise on the measured stress values is ignored it can be seen that observed and computed values are qualitatively similar on the time scale.

Fig 11 (a) gives a measured stress-strain curve for the chaser head at a point immediately beneath the anvil. The circuits for these gauges were independent of those used on the pile and are comparatively free of noise. Only the first 60 ms of time are given so reflections from the pile tip do not affect the results. Hence the conventional and modified wave equation results are identical over this period. Records of this type have been found invaluable for indicating the efficiency of a blow and the action of the cushion and anvil in transmitting the stresses to the head of the chaser. For the Menck 7000 peak driving stresses were typically of the order 22000 lbs/sq in and the operating efficiency 70%.

Pressure measurements

All six pressure cells continued to operate satisfactorily until the end of the test, but abrasion of the cable led to erratic records particularly during the redrive. A temporary loss of contact occurred with the upper pore pressure cell during the main drives but this was re-established when the cells were linked to the small data logger.
The changes in pressure recorded during the three main drives are shown in Fig. 12 against depth of the sensor below mudline. These results are displayed again in Fig. 13 and Fig. 14. The pore pressures are compared to the hydrostatic water pressure, and the total pressure to the in-situ total pressure calculated by assuming that the coefficient of earth pressure at rest (K₀) is equal to 0.7. The excess pressures generated agree sensibly with those that would be predicted theoretically (Ref.3).

Figure 15 shows the changes in total pressure and pore pressure that were recorded during the 'set up' period. Readings ceased after 9 days because the supply voltage became too low. It can be seen that the nine days represented only a small proportion of the time required for all excess pore pressures generated during driving to dissipate.

The changes in pressure recorded during the re-drive were not significant except at the lowest level of cells where both pore pressure and total pressure increased by around 80 psi.

HIGH SPEED PHOTOGRAPHY

During the main drives four films each 61m (200 ft) long were exposed at a speed of 200 frames per second. These films recorded the Menck 3000 and the Menck 7000 hammers operating in the cold and in the hot states. The quality of the film was a function of the ambient lighting conditions. The two films of the Menck 3000 proved to be of good quality but those for the Menck 7000 could not be analysed. The films have been examined using a travelling microscope. Two successive blows have been analysed from each of the Menck 3000 films.

Due to the poor quality of the M7000 films taken during the driving of another pile at this location has been analysed. Four blows have again been examined, two successive ones on the cold hammer and two on the hot hammer.

The results indicate that efficiencies for the M3000 and the M7000 based upon hammer impact velocity were in the ranges 61 to 67% and 62 to 95% respectively.

CONCLUSIONS

Due to the constraints on the length of this paper it has proved impossible to fully discuss the results of the study here. It is intended to present a more complete appraisal of the data obtained in a future publication.

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REFERENCES


APPENDIX I

METHOD FOR INCLUSION OF GRAVITY AND ROCKWELL CONNECTOR IN WAVE EQUATION ANALYSIS

Depending upon the chaser configuration used during driving, the pile make-up has either one or two gravity connectors. The chasers are of considerable length (see Fig. 1). They consist of lengths of pile joined by Rockwell connectors.

At any cross-section containing a connector relative movement of the cross-section is possible. This introduces an extra degree of freedom into the system. Account must be taken of this if acceleration and stress records are to be properly interpreted. The standard wave equation analysis Ref 3 has been modified to include these features.

The action of a gravity connector is such that when the pile segment is in compression, structural integrity is maintained and when the force is zero it is lost.

The Rockwell connector is a breech block connection. Due to wear, a small relative movement (δ) is possible between the threads as indicated diagrammatically in Fig. A1. The movement is clearly visible on high speed films of connectors taken during driving. Due to friction, an axial force R₀ is required to cause slip. The structural idealisation of the connector (Fig. A2) is such that when the pile segment is in a state of either tension or compression less than R₀ no slip occurs. However, when the pile force equals R₀ slip may take place. The force-displacement relation for the segment is therefore as indicated in Fig. A3. Prior to a blow being struck the gravity connector will be closed, but will open under the action of the reflected wave, and may well open and shut several times after each blow.
Fig. 1 - Pile, chaser and guide configuration.
Fig. 2 - Soil borings and cohesion profiles for two typical boreholes.
Fig. 3 - Instrument locations.
<table>
<thead>
<tr>
<th>Drive No.</th>
<th>Hammer</th>
<th>Driving Date and Time</th>
<th>Set-up Time (hrs)</th>
<th>Total Chaser Length (ft)</th>
<th>Gravity Connectors</th>
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<td>N 3000</td>
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<td>288.5</td>
<td>1 x 13T</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>60.0</td>
<td>1 x 32T (Solid)</td>
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<td>2 x 13T (Hollow)</td>
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</tbody>
</table>

Fig. 4 - Pile driving record.
Fig. 5 - Range of penetrations over which measurable signals were obtained.
**Fig. 6** - Measured acceleration at 317 ft, 160 ft, and 55 ft levels.

**Fig. 7** - a) Measured stress values. b) Predicted stress values. c) Predicted stress values.
Fig. 8 - Idealised soil deformation model.

Fig. 9 - Predicted acceleration at 317 ft, 160 ft, and 55 ft levels.
Fig. 10 - Predicted acceleration at 317 ft, 160 ft, and 55 ft levels.

Fig. 11 - A) Measured stress values  B) Predicted stress values  C) Predicted stress values.
Fig. 12 - Pressures recorded during driving relative to position of cells below mudline.

Fig. 13 - Pore pressures during driving compared to hydrostatic pressure.
FIG. 14 - TOTAL PRESSURES DURING DRIVING COMPARED TO ASSUMED INITIAL IN-SITU TOTAL-STRESS.

FIG. 15 - VARIATION OF PRESSURES WITH TIME AFTER DRIVING.
Fig. A1 - Pile segment with rockwell connector.

Fig. A2 - Structural idealization as used in wave equation analysis.

Fig. A3 - Force displacement curve.
Practical Planning for Driving Offshore Pipe Piles

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ABSTRACT

Difficulties in driving offshore piles to design penetrations are not infrequent and result in costly delays. Some of the more frequent problems are illustrated by six case histories of long piles driven to penetrations from 200 to 300 ft in various offshore regions. Drivability of the installed piles for each of these cases was investigated in retrospect by the one-dimensional wave equation, a technique that can also be applied as an important step in preconstruction planning. The need for supplemental procedures to aid pile driving because of unfavorable installation conditions can frequently be anticipated from combined evaluation of wave equation results and practical pile driving experience. Supplemental pile installation procedures such as jetting and pre-drilling should be performed under close engineering supervision to assure that the completed pile installation is consistent with design requirements.

References and illustrations at end of paper.

INTRODUCTION

Fixed platforms have been built in waters as deep as 400 ft for drilling and producing offshore oil and gas fields, and in shallower waters of 50 to 100 ft for lighthouse structures and deep-sea terminals. Most of the platforms are of the template-type construction that is shown in Fig. 1(a). A tubular steel space frame or substructure is placed on the ocean floor, and then open-end pipe piles are driven through the main vertical or slightly battered members into the foundation soils. The piles are typically 30-in. to 48-in. in diameter. Another type of platform that has been used in regions where structures are exposed to severe storms, or to strong currents, or to large ice loads is illustrated in Fig. 1(b). These tower platforms have relatively few large diameter legs and incorporate a circular group of piles. The open-end pipe piles are driven through pipe sleeves attached either internally or externally to the large diameter legs.
In addition to axial operating loads, offshore structures are subjected to large horizontal loads from waves and currents, and possibly from ice and earthquakes. Under combined vertical and horizontal loading conditions, piles in deep water requiring ultimate compressive and tensile capacities of up to 3500 and 2000 tons, respectively, are installed to penetrations of about 200 to 350 ft. Penetrations of about 400 ft have been required at some locations, and total pile lengths have been almost 800 ft.

In the marine construction industry, difficulties in driving piles to design penetrations are frequently experienced and can create uncertainty in axial pile capacities and possible deficiency in flexural capacity. Efforts to deal with driving problems when they arise are time consuming and therefore, costly because of expensive derrick barge rentals. If procedures to aid pile driving are improperly applied, then such procedures may lead to questionable adequacy of the platform foundation.

Planning for driving offshore piles utilizing the one-dimensional wave theory is outlined in this paper, and supplemental installation procedures to aid driving are described. The effectiveness and the limitations of pile-driving systems and supplemental installation procedures are illustrated by six case histories.

PRECONSTRUCTION PLANNING

Foundation Boring. Foundation design and construction planning for a fixed platform should begin with the drilling of a deep-penetration soil boring. The boring should be drilled to a penetration greater than the anticipated penetration of the piles and within about 100 ft of the platform location to provide reliable information on soil stratigraphy and strength.

Structural Design Requirements. Piles to support offshore structures must have sufficient penetration below the mudline to carry the anticipated compressive and tensile loads with an adequate factor of safety. Required pile penetration is usually predetermined by the static method of analyzing soil conditions disclosed by the foundation boring(1, 4).

The pile wall thickness must be adequate to resist the combined flexural and axial stresses produced by vertical and horizontal design loads. High flexural stresses develop in a zone that extends from a short distance about the mudline to a depth of generally 40 to 80 ft below the mudline, resulting in a thick-wall section in that zone. If pile driving is terminated before design penetration is reached, the thick-wall section will be displaced above its intended position, and the pile may be overstressed when exposed to storm loadings.

An extreme situation may utilize a composite pile when it is anticipated that piling cannot be driven to required penetrations due to the dense sands or strong clay soils extending from near the seafloor. The capacity of a composite pile is the sum of the computed capacity of the initial larger diameter pile and the capacity of the insert pile developed below the tip of the initial pile section. Both members of the composite pile are taken into account in designing wall thickness to resist flexural stresses.

Wave Equation Analysis. The most common rational method for studying the dynamic performance of a hammer-pile-soil system is the one-dimensional wave theory(6) utilizing a digital computer. The pile and hammer are modeled by a series of concentrated masses which are connected to weightless springs. Field test correlations and parameter studies(5) reveal that the frictional soil resistance on the side of the pile should be represented by a series of damped side springs. The point resistance of the pile is represented by a single spring and dashpot. Representative data can be included in the computer program to model driving accessories such as a capblock, pile cap and cushion. The more common hammers with rated energies of at least 60,000 ft-lb used to drive offshore piles are listed in Table 1.

The wave equation provides an analytical method to predict the driving stresses induced in the pile and to evaluate the drivability of a hammer-pile system for the soil conditions at a site. In addition, pile driving accessories such as cushions, capblocks, and pile caps can be selected from wave equation results to optimize the energy delivered on the top of a pile. Results of wave equation
analyses are generally presented in the form of a curve of hammer blows per unit of penetration versus total soil resistance to driving, $R_u$. The ability of a selected hammer to drive a specific pile to its design penetration into dense sand and hard clay generally requires that the computed maximum soil resistance, $R_u$, be greater than the desired ultimate static capacity, $Q$, of the pile. In normally consolidated clays and sensitive overconsolidated clays, the resistance to driving is generally less than the static capacity because of the temporary loss of adhesion along the pile wall due to remolding of the soil during driving.

Assumptions made in the wave equation analyses of the case histories described in the subsequent section of this paper are given on Fig. 2. These include the viscous damping constants for the springs representing soil resistance along the side and at the end of the pile, the value of quake or the elastic deformation of the soil, and the distribution of soil resistance along the side of the pile. Wave equation solutions yield the absolute refusal to driving. However, a point is reached when further driving to achieve deeper penetration becomes uneconomical, and a value of 30 blows per inch has been chosen as a practical refusal to driving.

Engineering Supervision. Difficulties and inconsistencies in pile driving are frequent and demonstrate that foundation engineering effort should extend beyond the design and construction planning phases into the actual construction phase at sea to assure that pile installation is consistent with soil criteria used in the pile design. Successful construction planning should consider the possible need of alternate supplemental procedures to aid pile driving. It is prudent that a person familiar with the foundation design and construction planning be present during the pile driving operations to evaluate and control supplemental pile installation procedures.

CASE HISTORIES

Six case histories of field pile installations are presented that illustrate relationships of pile drivability and soil conditions. These case histories represent a range of pile-hammer-soil combinations from different geographic areas of offshore construction. For each case, pile driving records are presented with brief descriptions of foundation conditions, and wave equation predictions are discussed with respect to actual pile drivability. Comments are made on the effectiveness and limitations of supplemental procedures that were used to aid driving in some of the cases.

Case 1. Pile driving records of two piles installed for a structure in the Gulf of Mexico are shown in Fig. 3(a). Subsurface materials consist of very soft-to-stiff clays to a penetration of about 156 ft and primarily of dense sandy silt below that penetration. The structural design incorporated 42-in.-diameter open-end pipe piles to support compressive design loads of 975 tons, and the foundation design study indicated that these piles should be driven to 235-ft penetration.

All four piles for this structure drove to practical refusal between 177- and 192-ft penetration with either a Vulcan 040 or 060 hammer. An attempt was made to achieve additional penetration of one pile by removing the soil plug using a water jet-aerialift and then redriving the pile. This procedure resulted in only 6 ft of additional penetration. An alternate procedure was then chosen in which the water jet without an airlift was operated below the pile tip to a nominal penetration of 215 ft. Following this procedure, Pile 2 was driven to 236-ft penetration with a 040 hammer. Initial redriving of Pile 1 created no additional penetration. After a second jetting to 215-ft penetration, Pile 1 was driven to 239-ft penetration with a 040 hammer.

Both piles experienced markedly reduced driving resistances due to jetting operations. In the case of Pile 1, the effects of jetting apparently penetrated well below the depth that jetting presumably extended and possibly below the final tip level as indicated by the low terminal driving resistance. Little additional skin friction and minimal end bearing capacity were realized on the bottom portion of Pile 1, in which case the adequacy of Pile 1 should be viewed with some uncertainty. The driving record for Pile 2 shows an increase in terminal driving resistance below
the level of jetting and indicates that the
effects of jetting did not extend below 215-ft
penetration. The capacity of Pile 2 was
probably unaffected by the jetting operations.
These data illustrate the random and unpre-
dictable effects of jetting ahead of the pile
tip. As in the case of Pile 1, the effect can
be detrimental.

Wave equation analyses were per-
formed for two pile-hammer-soil conditions
at this site, and the results are shown in
Fig. 3(b). The analysis for the 060 hammer
represents the condition of the piles driven
to refusal in the sandy silt at 180-ft penetra-
tion. This analysis indicates that the max-
imum soil resistance, $R_u$ (max.), that the
060 hammer could overcome at practical
refusal (30 blows per in.) was about 1700
kips. The computed static compressive
capacity, $Q$, at 180-ft penetration was 2200
kips. The soil resistance during the driving
of a pile into sand may be greater than $Q$, the static capacity of the pile (4). Since the
wave equation analysis indicates that the
driving limit of the hammer, $R_u$ (max.), was
less than $Q$, the pile might have been expected
to reach refusal as the driving records indi-
cate. If such information on possible pile
refusal is developed during preconstruction
planning, it enables the contractor to choose
an alternate installation method and have the
necessary materials and equipment available
at the construction site if and when driving
refusal occurs.

The analysis of the 020 hammer repre-
sents a pile embedded entirely in soft-to-stiff
clay. The practical driving limit, $R_u$ (max.), is
1150 kips, and the computed static capacity
is 1000 kips. Based on the blow count data
in Fig. 3(a), soil resistance during driving
is 500 kips or one-half the static capacity.
This indicates that considerable remolding
of the clay soils along the pile occurred dur-
ing pile driving.

Case 2. Driving records for two
36-in.-diameter pipe piles installed for a
platform in the Gulf of Mexico are shown in
Fig. 4(a). The subsurface materials consist
entirely of very soft-to-very stiff clays with
exception of a thin sand layer at about 40-ft
penetration. During the installation of piles,
the contractor was interested in comparing
the performance of a diesel and a steam
hammer of comparable energy. The diesel
hammer was a Delmag D-44 with a rated
energy of 87,000 ft-lb, and the steam ham-
mer was a MRT S-20 with a rated energy of
60,000 ft-lb. No capblock was used with
either hammer.

The driving records clearly indicate
that the driving resistance in blows per foot
with a diesel hammer was almost 100 percent
greater than that for the steam hammer. In
addition, the pile driven with the diesel ham-
mer reached refusal at 210-ft penetration
whereas the pile driven with the steam ham-
mer was driven to design penetration. The
steam hammer was also able to drive the
pile that had reached refusal with the diesel
hammer to design penetration.

Results of wave equation analysis for
each hammer with the pile tip at 210-ft pene-
tration are shown in Fig. 4(b). These results
indicate that the diesel hammer is a superior
hammer with a driving limit, $R_u$ (max.), of
1680 kips as opposed to 1370 kips for the
steam hammer. The difference in predicted
performance might be attributed to two fac-
tors. The diesel hammer might not have
operated as efficiently as assumed in the
wave equation computations. The other pos-
sible factor is that the two hammers,
especially the diesel, were not modeled cor-
correctly in the wave equation solution.

Case 3. Poor hammer efficiency is
one of the major causes of pile driving prob-
lems. The pile driving records shown in
Fig. 5(a) illustrate the effect of variable
hammer efficiency on the driving of two
typical 36-in.-diameter open-end pipe piles
installed for a structure off the Louisiana
cost. The subsurface materials consist of
very soft-to-firm clay from the seafloor to
195-ft penetration, silty sand from 195- to
255-ft penetration, and interbedded clay
and sand below 255 ft. The piles were
driven with two MRT S-20 hammers. Pile 1
was driven to practical refusal at 208-ft
penetration with Hammer 1. At this point, a
second hammer was brought to the job. The
driving resistance dropped drastically with
Hammer 2, and the pile was driven easily to
design penetration. After Pile 2 was driven
to 136-ft penetration with Hammer 1, Ham-
mer 2 again was substituted with similar
results. A comparison of driving records
for Piles 1 and 2 between 136- and 208-ft penetration shows the harder driving required with Hammer 1 due to hammer inefficiency.

The piles were designed to terminate at 235-ft penetration which is within the sand stratum. When the driving resistance dropped drastically with Hammer 2, it was concluded that the sand stratum was not of such a gradation and density that it contributed high end bearing support to a pile. Therefore, the design penetration was increased to 300 ft to achieve the required capacity. It is significant to note that the refusal of Pile 1 under Hammer 1 at 208-ft penetration would have been misinterpreted as indicating a high end bearing contribution from the sand layer if the improper functioning of Hammer 1 had not been revealed by the superior performance of Hammer 2.

A wave equation analysis was performed to determine the driving limit of the MKT S-20 hammer with the pile tip positioned at 220-ft penetration within the stratum of silty sand. The analysis shown in Fig. 5(b) indicates that the driving limit of the hammer, \( R_u \) (max.), is 1470 kips and that the resistance during driving was about 870 kips. The computed static capacity, \( Q \), is about 2000 kips. It might have been anticipated that piles would reach refusal after entering the sand stratum since \( R_u \) (max.) is less than \( Q \). However, this case shows that density and gradation can affect the soil resistance during driving.

Case 4. Pile driving records for two 30-in. piles driven for a structure off the coast of Nigeria are shown in Fig. 6(a). Subsurface materials at this site consist primarily of firm-to-very stiff clay to about 170-ft penetration with dense sand below that penetration. To achieve design capacity, it was necessary that the piles be driven to firm bearing in the deep sand. The last field splice to the piles was made after the piles had been driven to about 130-ft penetration. For seven of the eight piles for this structure, the splice required 3.5 to 4.5 hr. As illustrated by Pile 1, there was a marked increase in driving resistance after the delay; however, all of these seven piles were successfully driven to design penetration at about 175 ft.

One of the piles, illustrated by Pile 2, encountered an 8.5 hr delay, and as a result of the greater delay, the pile failed to move under 800 blows of the hammer. It also failed to move after soil plug removal and 550 subsequent blows.

For the type of soil conditions present at this location, gain in soil resistance with time due to setup or thixotropic effects in the clays can cause driving difficulties if lengthy delays occur after a pile penetration has been reached where the hammer capacity is almost fully utilized. This case illustrates that:
(1) final splices for piles driven primarily in soft-to-stiff clay soils should be made with the pile tips at the highest possible elevation;
(2) piles should be redriven as soon as possible after splices have been made; and
(3) soil plug removal to aid driving of a pile which has refused in a clay layer is of little benefit.

The occurrence of setup due to unavoidable delay may not be a problem if the proper hammer-pile combination is used. The wave equation was used to analyze two different hammer-pile combinations, and the results are shown in Fig. 6(b). Analysis of a pile section with 0.5-in. wall, corresponding to the piles installed in Case 4, indicated a driving limit, \( R_u \) (max.), of 880 kips. The computed static capacity, \( Q \), of the pile at 130-ft penetration was 930 kips. At the time driving was stopped to make the last field splice, the wave equation analysis indicates a soil resistance of 500 kips. As a result of setup after driving stopped, the soil resistance increased from 500 kips and approached the static capacity, \( Q \), of 930 kips. It is apparent that during the 8.5 hr delay the soil resistance increased to the driving limit of 880 kips and resulted in the refusal.

The other analysis illustrated in Fig. 6(b) is for the same pile-hammer combination except that the pile has a wall thickness of 1 in. The results indicate an increase in the driving limit to 1320 kips due to added wall thickness. Even if the soil resistance during the delay would have increased to the static capacity, \( Q \), the driving limit would have been 1.4 times greater. There are of course other pile-hammer combinations that could be determined by the wave equation to prevent refusal due to setup.
Case 5. Pile driving records for three typical 36-in. piles successfully installed at a platform in the Gulf of Mexico are shown in Fig. 7(a). Subsurface materials consist primarily of soft-to-very stiff clays with a stratum of sandy silt present between 161- and 181-ft penetration. When driving the first pile for the structure, Pile 1, the final 5/8-in. -wall addon section flared or bulged just above the jacket leg after it was driven about 5 ft with an S-20 hammer operating at a rate of 60 blows per min. The bulge was on the top side of the pile, which had a true batter of 8.5-vertical to 1-horizontal, and was positioned just above the stabbing guide. The length of the last addon was approximately 90 ft. The damaged portion was removed, and the pile was redriven with Hammer 2, also an S-20 hammer, operating at 43 to 50 blows per min. The pile did not flare again, but it twice reached refusal in the stratum of sandy silt and each time required soil plug removal before driving could be resumed.

Because of the driving difficulties with Pile 1, 7/8-in. -wall pipe was substituted for the 5/8-in. -wall pipe as the last addon for the remaining piles. The last addon section for Pile 2 was also driven with Hammer 2, but it was repaired after driving Pile 1. Pile 2 met refusal once in the stratum of sandy silt, but was readily driven to design penetration after the soil plug was removed. The driving resistance with Hammer 2 was noticeably lower after repair. Pile 3 was driven with Hammer 1 to design penetration without incident or jetting.

This case illustrates that: (1) continual hammer maintenance is necessary to keep a hammer operating at a high efficiency; (2) piles can be overstressed during driving even though wall thickness is adequate for the combined stresses due to maximum structural loads; and (3) soil plug removal is an effective aid to driving if tip resistance is a major portion of the total soil resistance during driving.

The wave equation analysis was utilized to determine if Pile 1 was overstressed by the stress wave induced by the hammer. The analysis did not indicate any overstressing due to driving alone. Combined stress including both driving effects and hammer and pile weight should be a maximum on the bottom side of a battered pile rather than on the top side. Other causes of the failure could have been the result of a steel deflect or the influence of the welds of the stabbing guide.

The wave equation was also used to determine if Piles 2 and 3 would drive more efficiently due to the thicker pile wall of the last addon. The results are shown in Fig. 7(b) and indicate that Piles 2 and 3 could overcome about 5 percent more soil resistance than Pile 1. This is only a nominal improvement in drivability, but in the case of Pile 3 it may have helped prevent refusal in the sandy silt stratum.

Case 6. Pile driving records for two typical piles installed for a structure in the Java Sea are shown in Fig. 8(a). Very soft-to-very stiff clay soils at the site are interbedded with three sand layers present between 78 and 91 ft, 99 and 121 ft, and 129 and 151 ft. A Vulcan 020 hammer was able to drive all piles through the first sand layer, but all piles reached refusal in the second sand layer. In Pile 1 the soil plug was then removed by air-lifting to within 1 ft of the pile tip, and the pile was redriven until it reached refusal in the third sand layer. The soil plug was again removed, and the pile was driven to design penetration.

After Pile 2 reached refusal in the second sand layer, the soil plug was removed, and a pilot hole was drilled about 20 ft below the pile tip using a power swivel and sea water. When redriven, the pile did not meet refusal in the third sand layer and reached design penetration without difficulty. The blow count did not increase significantly in the third sand layer, possibly indicating that an oversized pilot hole had been formed by drilling through that layer. Since the sand layer was thin in comparison with the total pile penetration, the overall effect on pile capacity was small.

The wave equation analysis for the case of the pile driven to refusal at 110-ft penetration in the second sand layer is shown in Fig. 8(b). The analysis indicates that the Vulcan 020 hammer should be able to overcome a soil resistance of about 1260 kips. The computed static capacity of the pile is 1100 kips and 110-ft penetration. Since the
piles met refusal, it appears that the soil resistance during driving into the sand stratum was greater than the computed static capacity.

PROCEDURES TO AID DRIVING

The most economical pile installation procedure is by driving alone without resorting to any supplemental procedures to aid the pile driving. However, the foregoing case histories illustrate that unfavorable soil conditions and driving equipment problems can prevent piles from being driven to desired penetrations although optimum utilization is made of available equipment and materials. Pile driving difficulties can frequently be anticipated from wave equation analyses and engineering experience so that planning of supplemental procedures can be made prior to mobilization of marine construction equipment.

Most driving difficulties are encountered with hard clays (shear strength greater than 2 tons per sq ft) and dense sands. Supplemental procedures that have been used successfully in such cases can be divided into three categories: (1) drilling and/or jetting to remove the soil plug inside a pile; (2) drilling an undersized pilot hole below the tip of the pile; and (3) driving an insert pile through the initial driven pile. The following paragraphs describe briefly current jetting and drilling techniques and the relative merit of each of the three supplemental procedures.

Removal of Soil Plug. Jetting or drilling to remove the soil plug inside a pile, when successful, is the most advantageous of the three methods. As shown by Fig. 9(a) and (b), the procedure destroys most of the end-bearing resistance of the pile and therefore is most effective when the point resistance, \( R_p \), is relatively large compared with the total soil resistance, \( R_u \). Removal of the soil plug when the pile tip is in clay is usually of minimal benefit, because the end resistance is generally in the order of 10 percent or less of the total driving resistance. An extreme example of this is illustrated by Pile 2 in Case 4 where setup increased the soil resistance to be overcome and the end resistance became proportionately less. In sands, the ratio of \( R_p/R_u \) can range from about 80 percent at shallow penetrations to about 50 percent at penetrations in the order of 250 ft. Plug removal is most effective when a pile is stopped in sand; however, the effectiveness becomes marginal in a deep-lying sand stratum or at deep penetrations into a thick sand stratum.

Jetting inside a pile can be accomplished by hanging 4-in. or larger drill pipe from the boom of a derrick barge to reach the pile tip and using a jet pump capable of delivering a large volume of water at a relatively high pressure. Removal of soil loosened by jetting inside a pile usually will require an airlift. If the internal soil plug consists of strong clays, then a combination of drilling and jetting using an air-water mixture is more effective and usually is accomplished by means of a hydraulic power swivel hung with the drill pipe from the derrick boom. To advance a pile tip through thick sand strata, it is sometimes necessary to repeat the plug removal-pile driving cycle one or more times.

If the design capacity of a pile includes end bearing in sand, then the pile should be cleaned out after reaching design penetration and the bottom of the pile grouted. The length of the grout plug must be sufficient to fully restore the ultimate end bearing of the pile.

Controlled Drilling Below Pile Tip. When removal of the internal soil plug proves ineffective and the pile continues to refuse, another supplemental procedure is to drill an undersized pilot hole to a selected depth below the pile tip and then to redrive the pile. This procedure is illustrated by Fig. 9(c) and its success relies on drilling a stable hole of uniform diameter, thereby reducing loss of soil-pile friction along the length of the pilot hole.

Uncontrolled procedures to make a hole below the tip of a pile were employed in Cases 1 and 6. Jetting in sand for the two piles reported in Case 1 effectively aided driving of the piles to grout, but the load capacity of one of the piles was severely affected. The wide difference in final driving resistance of the two piles in Case 1 illustrates one of the greatest difficulties with jetting ahead of a pile tip, namely, that the resulting effect on pile capacity is largely unpredictable and uncontrollable. In Case 6, uncontrolled
drilling was done to a depth of about 20 ft below the tip of a pile to aid driving the pile through the basal portion of a sand stratum and through a lower sand stratum. The low driving resistance through the lower sand indicated that the formation was disturbed; however, the zone was thin in comparison with the total pile penetration so that the overall effect on pile capacity was small.

Controlled drilling of a stable, uniform pilot hole can be accomplished with greatest assurance by using bentonite drilling mud and centralizers in a stiff drill string to produce a straight hole. The diameter of pilot holes in clay formations should be no more than about 75 percent of the outside diameter of the driven pile and at least 6 in. less than the pile diameter. A lesser diameter of about 50 percent of the pile diameter should be used in sands. With these precautions possible softening or sloughing of the sides of the hole beyond the limits of the pile will be minimized, and friction along the pile wall will be more nearly equivalent to that of a pile installed entirely by driving.

Pilot holes should be drilled under continual engineering supervision to control and evaluate the drilling method for maintaining a stable hole. Especially in sands, lateral yielding of the hole wall must be minimized to avoid consequent loss of skin friction along the pile driven through the length of the pilot hole. Drilling a uniform diameter straight pilot hole depends largely on the initial alignment of the hole below the pile tip. Under normal drilling conditions, an initial straight hole will tend to remain straight. To produce an initial straight hole, centralizers should be placed in a relatively rigid drill string to avoid wobbling of the drill rods inside the driven pile. Prepared bentonite drilling mud should be used to minimize scouring of the hole wall by the return drilling fluid and sloughing of the sand walls. The annular velocity of the return drilling mud must be kept low, and the mud viscosity should be relatively high to lift the sand from the hole. The drilling mud should form a sound wall cake to prevent filtration of drilling fluid into a sand formation that may cause weakening and sloughing of a hole. The pile should be driven to the level of the bottom of the pilot hole or lower as soon as drilling is finished to minimize the period that the muddied hole remains open.

Pilot holes should be terminated, in our opinion, at least 10 to 15 ft above design penetration to minimize disturbance of the soils that will provide end bearing resistance, and the pilot hole length should be less than the length of pile remaining to be driven before the next addon section must be made. With experience gained at a given site, adjustment of the pilot hole depth can be made to reduce the driving resistance to an acceptable range. For vertical piles, the maximum depth of holes drilled under close engineering supervision should desirably not exceed 40 to 50 ft. With steeply battered piles, the hole length should probably be restricted to about 25 ft to minimize adverse affects of deviation of the driven pile from its intended concentric position with the drilled hole. If any portion of the pile wall should be within the pilot hole, then the skin friction on that portion of the pile will be considerably less than the friction associated with an undisturbed formation.

More than one repetition of the drilling-driving cycle may be necessary to drive a pile to its desired penetration. Consideration should be given to restoration of the end bearing of a pile installed by drilling an undersized pilot hole. A positive method is to clean out the inside of the pile to within 5 to 10 ft of the pile tip and place a grout plug of sufficient length that may vary from 20 to 50 ft in sand.

Insert Piles. Insert piles provide a useful backup procedure when soil plug removal and drilling of pilot holes are ineffective because of the unfavorable soil conditions, or when a pile fails to encounter an expected bearing stratum. Fig. 9(d) is a sketch of an insert pile. The thick-wall section of the pile that was unable to be driven to grade in Case 4 was about 50 ft above its intended position at the mudline. The deficiency of pile wall thickness was rectified by installing the 24-in. insert pile, and the axial pile capacity was increased by driving the insert pile through the clay and into the sand where the other piles were terminated. The upper end of the insert pile was welded to the top of the larger pile, and the annular space between the two piles was grouted.
The insert pile is driven after removal of the soil inside the pile and penetrates below the tip of the larger pile. The problem remains, particularly in sand formations, whether the insert pile can be driven deep enough for the composite pile to develop the required compressive and tensile capacities. Drivability of an insert pile can be optimized by selecting an appropriate wall thickness from a wave equation analysis for the available driving equipment.

### PILE HAMMER EFFICIENCY

Selection of an optimum pile-driving system based on wave equation analyses or on engineering judgment assumes that the pile driving hammer will operate efficiently and deliver the manufacturer's rated energy to the head of the pile. Unfortunately, high-energy pile hammers used offshore are vulnerable to loss of efficiency during the severe service conditions that characterize most offshore construction. Four of the six case histories previously described include hammer problems to some degree. In Cases 3 and 5, the hammers were operating inefficiently causing considerable delays in pile installation. While hammer malfunction in many cases is quite evident, in some cases it is not, and even highly experienced construction personnel may not be aware of lowered efficiency when it occurs. Relatively rugged and simple instrumentation is needed to monitor the actual energy delivered to a pile so that driving problems and terminal driving resistances can be evaluated more effectively and confidently.

Instrumentation has been developed and used for land construction to measure the actual energy delivered to the top of pipe piles. Lightweight flat strain transducers bolted to the exterior of steel pipe piles have been used successfully to measure the driving force. Pile head deflection has been measured with linear variable displacement transducers, and heavy duty models are now available with a linear range of 16 in. Relatively compact and portable electronic equipment has been developed for recording field data from these devices. For offshore application of this instrumentation, the devices might be attached to the pile cap when the need arises to check hammer efficiency. Another procedure to determine hammer efficiency is to measure the velocity of the hammer ram, either by high-speed photography or by radar techniques. The instrumentation must yield a quantitative measure of hammer efficiency.

### SUMMARY AND CONCLUSIONS

Proper construction planning of pile driving for an offshore platform requires selection of an adequate driving hammer, of suitable driving accessories, and of an optimum pile configuration. The one-dimensional wave theory provides one useful tool to investigate pile drivability. When this technique is combined with sound foundation engineering judgment, it can be used to select the pile-driving system and to anticipate the need for supplemental procedures to aid driving.

The six case histories described in the paper lead to the following conclusions:

1. With available hammers, properly sized piles can be driven to design penetrations into normally consolidated clays and moderately sensitive overconsolidated clays without undue difficulties. However, if pile driving operations should be delayed for a significant period, then the gain in clay shear strength, or pile setup, can be sufficient to prevent further driving of the pile.

2. Sand strata are frequent obstacles to driving of piles, because pile end bearing is proportionally large in relation to the total soil resistance to driving. To achieve penetration through dense clean sand strata, experience indicates that the hammer-pile system should be capable of overcoming a soil resistance during driving that is greater than the computed static capacity.

3. The most convenient aid to driving through sand and hard clay is removing the internal soil plug down to the pile tip and then redrivering the pile. A more effective procedure is controlled drilling below the pile tip. This procedure is preferable to jetting below the pile tip, but its use carries some risk of damage to the pile load capacity and should be used with great caution. The pilot hole should be stabilized with prepared drilling mud, and its diameter should be uniform and as small as possible to achieve the desired purpose. A useful backup procedure in clay and sand is to drive an insert pile through the initial driven pile.
4. Loss of hammer efficiency during driving operations is a frequent occurrence but is not always readily detectable. Prompt repair or replacement of inefficient hammers is important in minimizing construction delays and costs and in optimizing pile drivability. Relatively simple and rugged devices are needed to measure the energy delivered by the hammer.

5. Pile driving operations and supplemental installation procedures should be performed under the control of an experienced foundation engineer to assure that pile installation is consistent with pile design.

ACKNOWLEDGEMENTS

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REFERENCES


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<td>80,000</td>
<td>3.0</td>
</tr>
<tr>
<td>VULCAN 560</td>
<td>STEAM</td>
<td>300,000</td>
<td>62,500</td>
<td>4.8</td>
</tr>
</tbody>
</table>

NOTE: ALL HAMMERS ARE SINGLE ACTING EXCEPT VULCAN 400C WHICH IS DIFFERENTIAL ACTING.
Fig. 1 - Types of fixed platforms.

\[ R_U = \text{Total soil resistance during driving} \]
\[ R_P = \text{Soil resistance at pile tip} \]
Viscous damping side, \( J' = 0.15 \text{ sec/ft} \)
end, \( J = 0.15 \text{ sec/ft} \)
\[ K_1 = \text{Cap block spring constant} \]

Fig. 2 - Assumptions for wave equation analysis.
Fig. 4 - Case 2: Comparison of steam and diesel hammers.
(a) Driving record

Pile: 36-in. O.D.
Hammer: MKT S-20

Pile 2 driven to 136-ft penetration with Hammer 1; Hammer 2 used below 136-ft

Very Soft to Firm Clay

Silty Sand

Interbedded Clay and Sand

(b) Wave equation analysis

Hammer efficiency = 70%
$R_U^{(max)} = 1470$ kips

Hammer efficiency = 40%
$R_U^{(max)} = 1120$ kips

Pile: 36-in. O.D
Hammer: MKT S-20
$K_t = 1 \times 10^4$ lb/in
(no cap block)

$R_p/R_y = 70$
$L_1 = 201$ ft
$L_2 = 220$ ft
$Q = 2000$ kips

Fig. 5 - Case 3: Hammer inefficiency.
Fig. 6 - Case 4: Pile setup.
Fig. 7 - Case 5: Successful installation achieved by improving hammer performance and removing soil plug.
(a) Driving record

Fig. 8 - Case 6: Uncontrolled drilling.
CASE HISTORIES - PILE DRIVING IN THE GULF OF MEXICO

by D.M. Stockard, Petro-Marine Engineering, Inc.

ABSTRACT

This paper presents a series of case histories on pile driving in the Gulf of Mexico, demonstrating the value of a pile drivability analysis to the engineer planning an offshore pile driving operation. In this study, actual pile-driving records are compared with the results of pile drivability analyses.

Piles in the study varied from 30 to 60 inches diameter with penetrations to 360 feet. Steamers with energy ratings of from 60,000 to 300,000 foot-pounds were used. Soils varied from under-consolidated clays with easy driving to dense sands through which driving was very difficult.

Also included in the studies is a simplified approach to pile drivability studies in which typical properties for soils, hammers, and cushions are used. In addition, various techniques are used to improve the probability of trouble-free pile driving. In each case, actual pile-driving records are compared with results of the pile drivability analysis.

INTRODUCTION

Considerable research has been done in the past concerning the dynamics of pile driving. Increasing demands of the present day offshore oil industry have necessitated construction of larger platforms in deeper water requiring larger piles with greater penetrations. As this occurs, the driving of the piles becomes more critical to the overall design.

Recent attention has been focused on pile driving in new exploration areas such as the North Sea, the Gulf of Alaska, or offshore California. The older, more active producing areas in the Gulf of Mexico have been largely overlooked. One of the aims of this paper is to demonstrate the usefulness and accuracy of pile drivability analyses in the Gulf of Mexico.

A pile drivability analysis provides significant insights during both the design and installation phases. Its primary purpose is to insure proper and efficient pile installation in the field. The analysis accomplishes this by aiding in the selection of proper hammer-cushion combinations, pile wall thicknesses, and add-on lengths for the particular site. The predicted blow counts from the analysis may be quite useful during pile installation in assessing hammer performance and actual soil conditions. This assessment allows the field engineer to determine any changes in equipment necessary during the pile driving operation.

BACKGROUND

The method of analysis used for pile drivability analyses is the one-dimensional wave equation, first proposed by Smith (1), and now generally used for dynamic analysis of pile driving. Later improvements were made by Samson, Hirsch, and Lowery (2) resulting in the Texas A & M Wave Equation Program (3). Further modifications in the last several years have resulted in more efficient versions of the program (4), but no analytical changes in the program have been made. The version of the program used for this study is the "TIDWAVE" program (August, 1975) (3).

BASIC TECHNIQUE

The model used for the one-dimensional wave equation idealizes the pile system as consisting of a ram, cushion, pile cap, pile, and surrounding soil. The physical and analytical model of a typical pile is shown in Figure 1. The pile hammer and pile are modeled as a system of concentrated masses connected by springs representing the stiffness of the pile and cushion. The soil is modeled by a spring and dashpot in parallel, attached to each concentrated pile mass below the mudline plus at the pile point.

The hammer and cushion properties used in the study are given in Table 1. The values shown are based on those recommended by researchers at Texas A & M (3). The factors of hammer efficiency and cushion stiffness have been found to be fairly typical. While the actual hammer used is noted on the driving record, the type of cushion is usually not known. For purposes of this study, when the actual cushion material was unknown, a wire rope cushion was assumed.
The soil parameters used with the model in this study are shown in Table 2 (4). The given values were used without variation in order to minimize the number of variables in the study.

The main variable in this study was the soil resistance at the time of driving. It has been generally assumed that the ultimate capacity of a pile driven in clay is 2 to 5 times the resistance at time of driving, while the ultimate capacity of a pile driven in sand is roughly equal to the driving resistance. Those multipliers are known as "setup" factors and take into account the increase in pile capacity from the time of driving to a period of several minutes to several months afterwards.

For purposes of this study, an upper and lower bound of expected soil resistance at time of driving were required. For an upper bound, the ultimate pile capacity attributed to clay was divided by a setup factor of 2.0, and the ultimate pile capacity attributed to sand was divided by a factor of 1.0. For a lower bound, the ultimate pile capacity due to clay was divided by a factor of 5.0, and that due to sand was divided by a factor of 1.2. The resulting envelope between the upper and lower bounds of resistance at time of driving is shown in Figure 2b for a typical soil offshore Louisiana.

It must be noted that many outside factors can influence soil resistance at time of driving. The given bounds of setup factors assume continuous driving only and do not take into account delays in driving such as pile add-ons, replacement of a cushion, hammer breakdowns, or weather problems. One reasonably good assumption shown by the results in this paper, is that when driving is resumed after a delay, the driving resistance is equal to the ultimate static capacity of the pile at that penetration and will not significantly be reduced for at least 30 feet in clay. Faster recovery normally occurs in sand. The results of delays due to pile add-ons at penetrations of 60, 135, and 193 feet are shown in the curve for the upper bound of driving in Figure 2d.

Once the model is defined and the soil resistance at time of driving estimated, the wave equation program is executed for the particular hammer-cushion combination and penetration and the results are plotted as blow count versus driving resistance. Figure 2c shows typical curves for varying penetrations.

Using the estimated driving resistances for various penetrations, such as in Figure 2b, and the blow-count-versus-driving-resistance curve, such as in Figure 2c, the blow count may be estimated for various penetrations. Using the upper and lower bounds of driving resistance, an envelope of predicted blow counts may be obtained as shown in Figure 2d.

CASE HISTORIES

For purposes of this study, case histories were grouped according to general locations:

1. Offshore Louisiana - South Timbalier, Ship Shoal, Eugene Island, South Marsh Island and Vermilion Areas.

2. Offshore Texas - High Island, Galveston, Brazos, Matagorda Island, and Mustang Island Areas.

Several cases are presented from each general area. For each case, soil data, drivability predictions and the actual driving record are given. The graphs of the driving record show the blow count as points for every pile, every 5 feet, and at add-ons. No piles are omitted. Every attempt has been made to show a cross-section of pile driving in the Gulf of Mexico.

Offshore Louisiana

The sites for studies in this group are located in the "belt" of platforms about forty miles off the Louisiana coast in the South Timbalier to Vermilion Areas. Soil conditions were fairly similar, although pile drivability was not.

Case 1:
The first case, appropriately, was one in which problems were encountered in driving. Piles 42 inches in diameter were to be driven to a penetration of 270 feet. The first 140 feet was firm to stiff clay and the remainder primarily fine sand (Figure 2a). As was the custom of many offshore operators, a drivability analysis was not considered necessary. As shown in Figure 2d, problems were encountered in driving through the sand at a penetration of 200 feet and jetting became necessary.

A drivability analysis was later performed to determine if the problems could have been predicted, thus allowing an opportunity to modify the pile or driving equipment to eliminate or minimize jetting. The soil skin friction (solid line) and end bearing (dashed line) are shown in Figure 2a. In Figure 2b are shown the ultimate static pile capacity and the upper and lower bounds of driving resistances. The method for arriving at the driving resistance was presented earlier in the paper.) Figure 2c shows the predicted blow count versus driving resistance curves obtained from three wave equation runs utilizing different penetrations and soil conditions. Figure 2d gives the resulting drivability prediction along with the actual driving record.

As can be seen, the actual blow counts agree generally very well with the predicted values, falling within the upper bounds of driving. The drivability analysis also indicated that the pile could not be driven beyond 200 feet penetration with the Vulcan 060 hammer, without jetting, as was the actual case. A drivability analysis performed beforehand would have indicated this problem.

Further studies showed that a Vulcan 060 hammer and increased pile wall thickness (from 0.875 to 1.375 inches) could have allowed the pile to be driven to a penetration of 240 feet. This is still short of the required penetration of 270 feet, but only 30 feet instead of 70 feet of jetting would have been required.

Case 2:
In a second case piles 48 inches in diameter were driven to a penetration of 325 feet. The soil at the site consisted of stiff-to-hard clay with a 15-foot sand layer at about 160 feet penetration. Skin friction and end bearing curves are given in Figure 3a.
Figure 3b shows the predicted drivability along with the actual driving records. A very good prediction was shown and the piles were driven with no problems using Vulcan 020, 040, and 060 hammers. If the Vulcan 060 hammer and a thicker minimum pile wall thickness (0.875 inch instead of API RP2A minimum of 0.750 inch) had not been used, the piles probably couldn't have been driven to design penetration.

There is one important point to note from Figure 3b. The blow counts at a penetration of 250 feet and greater did not increase as expected using a clay setup factor of 2.0 to 3.0. The final blow count corresponded to a clay setup factor of 3.5, which is higher than normal, but still within the generally accepted range of 2.0 to 5.0.

Case 3:
A third case offshore Louisiana involved piles' 42 inches in diameter driven to a penetration of 280 feet using Vulcan 040 and 060 hammers. The soil at the site consisted of layers of clay, sand, and silt as shown in Figure 4a.

A drivability analysis was made before the installation and was later refined. Figure 4b shows the results of the drivability predictions and the actual blow counts. Increased blow counts were predicted for pile add-ons at penetrations of about 125 feet and 215 feet. The record for the add-on at 125 feet indicates much more setup had occurred than expected. The driving resistance would have to be about 50 percent greater than the ultimate capacity at 125 feet to explain the actual blow counts. Other factors, such as reduced hammer efficiency, may have caused the large increase in actual blow counts in the field. Otherwise, the actual blow counts were well within the predicted limits.

Case 4:
A fourth case offshore Louisiana involved piles' 48 inches in diameter to be driven to a penetration of 300 feet. The soil consisted entirely of soft to stiff gray clay (Figure 5a).

A drivability analysis was conducted in order to improve the probability of driving the piles to final penetration with a Vulcan 060 hammer. The wall thicknesses were adjusted, allowing a minimum thickness of 0.875 inch below the mudline, instead of the 0.750 inch minimum wall thickness according to the API RP2A. The installation contractor chose to use Vulcan 360 and 560 hammers for the job. As a result, the actual blow counts were much lower than expected with the Vulcan 060 and a hindcast was made using the Vulcan 560 (Figure 5b).

The actual driving record is also shown in Figure 5b. The predictions matched very well except for two piles whose driving resistance dropped off more slowly after the second add-on, at about 180 feet penetration. Once again, the unknown factor of setup after a delay in driving may have affected driving resistance.

Offshore Texas-Louisiana

A second general area of the Gulf of Mexico is the oil and gas fields offshore the Texas-Louisiana border in the South Additions of the East Cameron, West Cameron, and High Island Areas. Soil conditions are generally different in those areas than closer to shore.

Case 1:
The best documented case available was for a site about 100 miles offshore. Forty-two inch diameter piles were driven to a penetration of 280 feet through soils consisting of very soft to stiff clay, silty fine sand, and then a thick layer of fine sand (as shown in Figures 6a and 6b). A problem might be expected in driving through the silty fine sand into the fine sand layer deep enough to get the required capacity.

A drivability analysis was made to determine if the piles could be driven and what hammer would be required. The results of the drivability analysis for different hammers, cushions, and strokes to be used are shown in Figure 6c. The resulting predicted drivability and actual blow counts are shown in Figure 6d.

The actual driving records matched the predicted values except for a few local problems, such as at 185 feet penetration when the Vulcan 360 hammer was being shut down for an add-on. The construction crew on the derrick barge was surprised at the accuracy of the prediction and how well hammers could be selected ahead of time to achieve good driving. Even the engineers were surprised when the design penetration was reached at a blow count of 80 blows per foot, 22 feet into the sand layer through which driving would normally be considered improbable. Once again, the drivability analysis was right on target in predicting efficient driving.

Offshore Texas

A third general area in the Gulf of Mexico of growing importance in recent years is the fields within 20 miles of the upper Texas coast in the High Island, Galveston, Brazos, Matagorda Island, and Mustang Island Areas. The soils in this area are somewhat different from that offshore most of Louisiana.

Case 1:
The first installation to be considered required 36-inch diameter piles driven to a penetration of 240 feet. The soil consisted of alternating layers of clay and sand, possibly resulting in some drivability problems. The piles were designed for Vulcan hammers, but a reanalysis was required for Menck 750 and 1800 hammers before installation of the platform.

The soil profile is shown in Figure 7a. Some problems were expected at the 150 foot level, so this penetration received special attention. The predicted blow counts, allowing for pile add-ons, are shown in Figure 7b, along with the actual pile driving records. A generally good comparison is seen with slight variations, probably caused by variations in the local sand layers at each particular pile. Once again, driving for these piles went very well, and jetting was avoided by an efficient pile make-up and hammer selection.

Case 2:
A second installation off the Texas coast was located about ten miles offshore. Thirty inch diameter piles were required to be driven 140 feet into a predominate stiff clay soil (Figure 8a). The piles were driven with a Conmaco 020 hammer, similar to a Vulcan 020 hammer, with an asbestos cushion. The recorded blow counts and results of a hindcast drivability analysis are shown in Figure 8b.
For shallow penetrations, the results agreed fairly well. After the pile add-on at about 85 feet penetration, the pile did not drive as well as predicted. The difference may have been caused by a stiffer soil than expected, inefficiency in the hammer, or possibly other unknown problems. This case shows that a drivability analysis is not always going to have perfect results since many unknown variables are always present.

Case 3:

A third platform was installed about ten miles off the coast. A Vulcan 020 hammer was used to drive 30-inch diameter piles to a penetration of 170 feet into a firm-to-very-stiff clay (Figure 9a).

Once again a drivability analysis was considered unnecessary and was only performed as a hindcast. The resulting predictions and actual blow counts compared in Figure 9b, show very good agreement.

CONCLUSIONS

In conclusion, pile drivability analyses using typical soil, hammer, and cushion properties are reasonably accurate for Gulf of Mexico conditions. Drivability analyses have been extremely useful in predicting pile driving problems before they actually occur offshore thus saving time and construction costs.

RECOMMENDATIONS FOR FUTURE STUDY

As this study demonstrates, there is much more to learn about pile driving, even in the Gulf of Mexico. A continuing effort is needed with more information flow between operating oil companies, engineering firms, and construction companies to produce more reliable data for publication and presentation. When such knowledge is compared and traded, more success can be expected to be achieved in this technical area.

ACKNOWLEDGEMENTS

The author expresses his thanks to Petro-Marine Engineering, Inc. for the support in the computer work, compilation, and presentation of this study. Thanks go to Fred Stelzer, Larry Bryant, Carol Frazier and Bob Belless for their review of this paper, to Elke Naumov for the drafting of the illustrations, and to Gerri White for the final typing of the paper. Much credit must be given to the design engineers at Petro-Marine who have performed the individual studies used as a preliminary basis for this paper.

Special thanks must also go to the clients of Petro-Marine whose data has been included. Special efforts have been made to insure confidentiality by withholding company identification and specific platform locations.

REFERENCES

Table 1: Hammer and Cushion Properties

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>RAM WEIGHT (lbs)</th>
<th>PILE CAP WEIGHT (lbs)</th>
<th>STRIKE (ft)</th>
<th>EFFICIENCY</th>
<th>CUSHION TYPE</th>
<th>STIFFNESS (kips)</th>
</tr>
</thead>
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<tr>
<td>VULCAN 120</td>
<td>20.0</td>
<td>10.0</td>
<td>5.0</td>
<td>0.80</td>
<td>ASBESTOS</td>
<td>9423</td>
</tr>
<tr>
<td>VULCAN 060/340</td>
<td>60.0</td>
<td>31.5</td>
<td>5.0</td>
<td>0.80</td>
<td>ASBESTOS</td>
<td>26242</td>
</tr>
<tr>
<td>VULCAN 060/350</td>
<td>60.0</td>
<td>35.5</td>
<td>5.0</td>
<td>0.80</td>
<td>ASBESTOS</td>
<td>32398</td>
</tr>
<tr>
<td>VULCAN 500</td>
<td>50.0</td>
<td>42.0</td>
<td>5.0</td>
<td>0.86</td>
<td>ASBESTOS</td>
<td>4300</td>
</tr>
<tr>
<td>HENC 750</td>
<td>58.58</td>
<td>22.05</td>
<td>4.92</td>
<td>0.75</td>
<td>BONGIASSI</td>
<td>24979</td>
</tr>
<tr>
<td>HENC 1800</td>
<td>58.58</td>
<td>22.05</td>
<td>4.92</td>
<td>0.75</td>
<td>BONGIASSI</td>
<td>40755</td>
</tr>
</tbody>
</table>

COEFFICIENTS OF RESTISTANCE:
ASBESTOS: 0.5; WIRE ROPE: 0.8; NICARTA: 0.8; BONGIASSI: 0.75

Fig. 1 - Hammer-pile-soil model.

Fig. 2a - Offshore Louisiana case 1 - soil data.

Fig. 2b - Offshore Louisiana case 1 - pile capacity and driving resistance.

Fig. 2c - Offshore Louisiana case 1 - wave equation results.
BLOW COUNT
(BLOWS / FOOT)

OFFSHORE LOUISIANA
CASE 1

PILE ADD-ON

UPPER BOUND OF DRIVING
VULCAN 040 HAMMER

LOWER BOUND OF DRIVING
PILE ADD-ON

DRIVING IMPOSSIBLE
WITH VULCAN 040
BEYOND 200 FEET

PENETRATION (FEET)

Fig. 2d - Offshore Louisiana case 1 - drivability prediction and driving records.

BLOW COUNT
(BLOWS / FOOT)

OFFSHORE LOUISIANA
CASE 2

PILE ADD-ON

VULCAN 020
VULCAN 040

UPPER BOUND OF DRIVING

LOWER BOUND OF DRIVING
PILE ADD-ON

PENETRATION (FEET)

Fig. 3b - Offshore Louisiana case 2 - drivability prediction and driving records.

SKIN FRICTION (KSF)

OFFSHORE LOUISIANA
CASE 2

Clay

Fine Sand

END BEARING (KSF)

Fig. 3a - Offshore Louisiana case 2 - soil data.

SKIN FRICTION (KSF)

OFFSHORE LOUISIANA
CASE 3

Soft Clay
Sand
Stiff Clay
Sandy Silt
Stiff to Very Stiff Clay
Silty Sand
Very Stiff Clay

END BEARING (KSF)

Fig. 4a - Offshore Louisiana case 3 - soil data.
Fig. 4b - Offshore Louisiana case 3 - drivability prediction and driving records.

Fig. 5a - Offshore Texas-Louisiana case 1 - soil data.

Fig. 6b - Offshore Texas-Louisiana case 1 - pile capacity and driving resistance.

Fig. 6c - Offshore Texas-Louisiana case 1 - wave equation results.
Fig. 6d - Offshore Texas-Louisiana case 1 - drivability prediction and driving records.

Fig. 5a - Offshore Louisiana case 4 - soil data.

Fig. 5b - Offshore Louisiana case 4 - drivability prediction and driving records.

Fig. 7a - Offshore Texas case 1 - soil data.
Fig. 7b - Offshore Texas case 1 - drivability prediction and driving records.

Fig. 8b - Offshore Texas case 2 - Drivability prediction and driving records.

Fig. 8a - Offshore Texas case 2 - soil data.

Fig. 9a - Offshore Texas case 3 - soil data.
Fig. 9b - Drivability prediction and driving records.
Evaluating Pile Drivability for Hard Clay, Very Dense Sand, and Rock


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Abstract

This paper presents case histories of 36 and 42-in.-diameter piles driven in the Arabian Gulf by hammers with rated energies ranging from 120,000 to 325,000 ft-lbs. Results of hindcast pile drivability studies are presented for sites where hammer performance was monitored using weldable strain gauges to record force-time histories. Procedures for computing the soil resistance during pile driving were determined by correlating field and predicted blow counts. Soil quake and damping parameters for the side and tip of the piles are recommended.

Introduction

Wave equation analysis of pile driving is based on the discrete element idealization of the hammer-pile-soil system formulated by Smith. One of the two major uncertainties in a wave equation analysis is the energy transmitted by the hammer to the pile-soil system. Field measurements have shown that the hammer efficiency is often significantly less than conventionally assumed. Cushion properties may vary significantly during the service life of the cushion. Bonglass hardwood cushions soften with use, as opposed to the stiffening tendency of wire rope. Hammer efficiency, and the stiffness and coefficient of restitution of the cushion were monitored during driving for the case histories presented in this paper. Monitoring hammer performance greatly enhances the value of a correlation study.

The soil-pile interaction during driving as described by the load-deformation behavior of the soil at the soil-pile interface and pile point is the other major uncertainty in a wave equation analysis. Smith defined soil-pile interaction in terms of soil quake and damping coefficients. These parameters are not intrinsic to soil properties, but rather coefficients that incorporate all that is not clearly understood about the process of driving a pile. Some of the data on which previous correlation studies have been based are for small diameter (12 to 18-in.) closed end piles, and may not be applicable to the large diameter open-end piles used offshore. Other studies performed for each driving may not be appropriate for References and illustrations at end of paper.

Hard driving because blow counts determined by wave equation analysis are insensitive to variations of soil and hammer properties at low blow counts.

Despite these uncertainties, wave equation analysis has proven to be a valuable tool for evaluating pile drivability when a database exists to define hammer and soil properties.

Driving Methodology

A pile drivability study consists of three parts. First, the driving resistance that can be overcome by a particular hammer-pile-soil system is computed from a wave equation analysis. Second, the soil resistance to driving is estimated from soil properties at the site. Third, these results are compared and an evaluation of pile drivability is made that should be tempered by judgment and past experience in the area.

The driving records of piles at a particular site often show considerable scatter because of variations in soil conditions, hammer efficiency, and cushion properties. Additional factors affecting drivability are clay setup during interruptions in driving and plug behavior. For these reasons, drivability studies should be used to predict a range in pile driving blow counts.

Wave Equation Analysis

The parameters selected for a wave equation analysis should model the actual hammer-pile-soil system as closely as possible. The hammer efficiency and cushion properties used in our analyses are measured values computed from recorded force-time histories. The quake and damping parameters recommended by Roussel were used in our wave equation analyses. These parameters were determined from a comprehensive correlation study performed for large diameter offshore piles, in which the driving records of 58 piles at 15 offshore sites in the Gulf of Mexico were analyzed. The side and point quake are assumed equal, with a magnitude of 0.10 in. for stiff to hard clay, silt and sand. Side damping in clay decreases with increasing shear strength, which is in agreement with the laboratory test results of Coyle and Gibson and Heerema. For a hard clay, side...
damping is 0.01 sec/ft and point damping is 0.15 sec/ft. For sand, side damping is 0.08 sec/ft and point damping is 0.15 sec/ft. Average values for side quake and side damping, weighted according to the relative contribution of each soil type to the shaft resistance during driving, were used in our analyses.

**ESTIMATE OF SOIL RESISTANCE TO DRIVING**

Computation of the soil resistance to pile driving is analogous to the computation of ultimate axial pile capacity by the static method. The resistance to driving is the sum of the shaft resistance and the point resistance. The shaft resistance is computed by multiplying the average unit skin friction during driving and the embedded surface area of the pile. The point resistance is computed by multiplying the unit end bearing and the end bearing area. Procedures used to compute unit skin friction and unit and bearing are discussed in the following paragraphs.

**Cohesive Soils**

For piles driven in cohesive soils, the unit skin friction during continuous driving is computed using the stress history approach presented by Scape and Gemeinhardt(5). The unit skin friction is first computed by the API RP 2A (January 1981) Method(6) in accordance with Sec. 2.6.4, Para. b.2. The unit skin friction is then adjusted incrementally using a pile capacity factor, Fp, determined empirically from wave analysis equations performed for six sites. The pile capacity factor is given by:

$$F_p = 0.5 \times (OCR)^{0.3}$$

The overconsolidation ratio (OCR) is estimated using the equation:

$$\frac{s_u}{s_{unc}} = (OCR)^{0.85}$$

where:

- $s_u$ = actual undrained shear strength of clay having a given PI, and
- $s_{unc}$ = undrained shear strength of the same clay if normally consolidated.

According to a relationship described by Skempton,

$$s_{unc} = \bar{\sigma}_v (0.11 + 0.0037 \ PI)$$

where:

- $\bar{\sigma}_v$ = effective overburden pressure, and
- PI = plasticity index.

Unit end bearing in clay is computed using the following equation:

$$q = \frac{s_u}{N_c}$$

where:

- $N_c$ = dimensionless bearing capacity factor.

A value of 9 is used for $N_c$.

**Granular Soils**

For piles driven in granular soils, the unit skin friction during continuous driving is computed using static pile capacity procedures, and is based on the equation:

$$f = K \bar{\sigma}_v \tan \delta$$

where $K$ = coefficient of lateral earth pressure, $\bar{\sigma}_v$ = effective overburden pressure, and $\delta$ = angle of friction between soil and pile.

The value of $K$ is taken as 0.7. The following table presents values of $\delta$ and $f_{max}$ used in our computation of unit skin friction.

<table>
<thead>
<tr>
<th>Soil Type</th>
<th>$\delta$, degrees</th>
<th>Friction Angle, $\delta$, degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sand</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Silty Sand</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Sandy Silt</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Silt</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

Unit end bearing in granular soils is computed using the following equation:

$$q = \frac{\bar{\sigma}_v N'}{N}$$

where $\bar{\sigma}_v$ = effective overburden pressure, and $N'$ = dimensionless bearing capacity factor.

The following table presents values of $N'$ and $q_{max}$ used in our computations of unit end bearing:

<table>
<thead>
<tr>
<th>Soil Type</th>
<th>$N'$</th>
<th>q_{max}, ksf</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sand</td>
<td>40</td>
<td>200</td>
</tr>
<tr>
<td>Silty Sand</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>Sandy Silt</td>
<td>12</td>
<td>60</td>
</tr>
<tr>
<td>Silt</td>
<td>8</td>
<td>40</td>
</tr>
</tbody>
</table>

For carbonate material, $\delta$ is decreased 5 degrees, and the values of $f_{max}$, $N'$, and $q_{max}$ for the corresponding soil type are used.

**Rock**

Driving piles into rock is anticipated to severely fracture the rock layers and reduce the rock to a granular material. Therefore, unit skin friction for piles driven in rock layers is computed assuming sand parameters. Silty sand parameters are assigned to rock layers that are interbedded with silt or clay seams and layers.

For poor to fair quality rock, unit end bearing is limited to values given for granular soils. For more competent rock, unit end bearing is computed using the following equation:

$$q = \frac{uN_u}{N}$$

where $u$ = compressive strength of rock, and $N_u$ = dimensionless bearing capacity factor.

A value of 3 is used for $N_u$.

Lower and upper bound values of soil resistance to driving are computed for both coring and plugged pile conditions. When a pile cores, relative movement between pile and soil occurs both on the outside and inside of the pile wall. Skin friction is, therefore, developed on both outside and inside pile wall. The end bearing area is equal to the cross-sectional area of steel at the pile tip. When a pile plugs, the soil plug moves with the pile during driving. Skin friction is mobilized only on the outer wall. The end bearing area is the gross area of the pile. Unlike static pile capacity computations, end bearing is not
limited to the frictional resistance developed by the soil plug.

The resistance to driving should be based on a reasonable upper bound in contrast to static pile capacity, which is based on a reasonable lower bound. When a pile cores, a lower bound value is computed assuming that the skin friction developed on the inside of the pile is 50 percent of that on the outside of the pile. For the upper bound coring case, the internal skin friction is assumed equal to the external friction.

When a pile plugs, a lower bound value of the resistance to driving is computed using the procedures presented in the preceding paragraphs to compute unit skin friction and unit end bearing. For the upper bound plugged case, values of unit skin friction and unit end bearing for granular soils are developed by increasing the lower bound values of unit skin friction by 30 percent, and increasing unit end bearing by 50 percent. A corresponding increase in limiting values for unit skin friction and unit end bearing is assumed for cohesive soils, and bearing is computed using Nc = 15, an increase of 67 percent. Unit skin friction is not increased.

CASE HISTORIES

The purpose of the case histories was to confirm the soil quake and damping parameters used in our wave equation analysis, and to determine the best procedure for estimating the soil resistance to driving. Our evaluation was made by comparing predicted and observed driving records. The estimated soil resistance to driving at a given penetration was used to determine the corresponding blow count from wave equation results obtained using the measured hammer performance data.

Unless otherwise stated, each case history is the installation of a six pile platform in the Arabian Gulf, consisting of 42-in. diameter piles having a constant 1.5-in. wall thickness, with a 5-ft long driving shoe having a wall thickness of 1.75 in. The case histories are divided into the following categories: very dense sand, hard clay, mixed profile, and rock.

VERY DENSE SAND PROFILES

Table counts generally increase with depth in most sand profiles encountered in the Arabian Gulf. Erratic counts with depth and between adjacent piles are partly a result of cemented sand seams and layers.

The soil stratigraphy and curves of estimated soil resistance to driving are shown in Figure 1 for a site in the Safaniya Field. A Menck 1800 hammer with a Bonnossi hardwood cushion was used to drive the piles through the very dense fine sand layer from 21 to 58-ft penetration. Shown in Figure 2 are values of hammer efficiency, cushion stiffness, and cushion coefficient of restitution measured during the driving of pile B-1. A decreasing cushion stiffness with increasing hammer blows is typical of a wood cushion. Use of an average value of stiffness in a wave equation analysis would underestimate blow counts and overestimate driving stresses when a worn cushion is used.

A comparison of observed and predicted blow counts for pile B-1 is presented in Figure 3. Predicted blow counts are badly underestimated using either the static pile capacity, or the soil resistance to driving for both the lower and upper bound coring cases. Good agreement is obtained for the lower and upper bound plugged cases. The driving records for all six piles are shown in Figure 4. Blow counts were computed using the measured hammer properties for each pile. The lowest blow counts predicted for the lower bound plugged case and the highest blow counts predicted for the upper bound plugged case are also plotted in Figure 4. Our predicted range in blow counts is in good agreement with the scatter observed in the field. The plugged cases give higher predicted blow counts than the coring cases not only because the soil resistance to driving is larger, but also because the percent tip resistance is larger.

A comparison of observed and predicted driving records for a second site in the Safaniya field is presented in Figures 5 and 6. A Vulcan 340 hammer with a steel plate and wire rope cushion was used to drive two piles to design penetration, and to initially drive the remaining four piles. Final driving of these piles was accomplished using a Vulcan 560 hammer. As was done previously, we have plotted the predicted blow counts determined from the hammer data giving the minimum value for the lower bound plugged case, and the maximum value for the upper bound plugged case. Good agreement is obtained assuming a plugged pile.

HARD CLAY PROFILES

Our experience with pile driving in hard clays in the Arabian Gulf has shown that blow counts may be constant with penetration, blow counts may be very erratic due to seams and thin layers of rock, and driving delays will result in an increase in blow counts during redriving.

The soil stratigraphy and curves of estimated soil resistance to driving are shown in Figure 7 for a site in the Zuluf Field. The pile capacity factor computed for this site is equivalent to a sensitivity of 2.18. The piles were driven through a hard calcareous clay stratum present from 26 to 69-ft penetration. The clay contains gypsum seams and layers at 43 to 46-ft and 54 to 55-ft penetrations, and numerous fragments of gypsum and claystone. Final pile driving was accomplished with a Vulcan 560 hammer with a cushion made up of alternating layers of 0.25-in.-thick steel plates and two layers of 0.25-in.-thick Ascon. Shown in Figure 8 are observed hammer performance data for pile B-1. The stiffness of the Ascon cushion has been found to be fairly constant until the Ascon discus disintegrate.

The curves of soil resistance to driving plotted in Figure 7 differ by only 33 percent between 42-ft and 64-ft penetration. Consequently, the range in predicted blow counts is a narrow band. Blow counts predicted for pile B-1 for the upper bound plugged case are roughly two blows per foot less than observed to 48-ft penetration, as shown in Figure 9. The driving records for all six piles presented in Figure 10 show considerable scatter. During the first few feet of redriving, blow counts are higher than predicted due to setup occurring when pile driving was interrupted to change hammers. Scattered high blow counts at deeper penetrations are due to the rock fragments, seams, and layers in the clay.
The field blow counts during final driving of a three-pile well protector in the Zuluf Field, plotted in Figure 11, are constant with depth with little scatter. Final driving of the 36-in.-diameter piles was accomplished with Vulcan 540 hammer and Ascon cushion. Increased blow counts at the completion of driving are due to a gypsum stratum encountered 5 ft higher than indicated by the boring. Good agreement is obtained with blow counts computed for the lower bound plugged case.

MIXED PROFILE

A platform installation in the Abu Safah Field was selected for our mixed profile. The soil conditions consist of a medium dense carbonate sand from 35-ft to 79-ft penetration, a very stiff to hard carbonate clay from 79-ft and 117-ft, dense carbonate silty sand from 117-ft to 131-ft, and a very stiff to hard calcareous silty clay from 131-ft to 147-ft. Except for the first pile section, piles were driven with a Menck 3000 hammer and a Bongasseri hardwood cushion.

Curves of estimated resistance to driving for both lower and upper bound coring and plugged cases are presented in Figure 12. Blow counts were computed at six penetrations. The coring and plugged cases give similar predictions, as shown in Figures 13 and 14.

ROCK

Our experience with pile installation in the Arabian Gulf has shown that piles may be driven through poor quality rock of weak strength using large hammers with rated energies greater than 300,000 ft-lbs. Blow counts exceeding 250 blows per foot have been required to drive piles through weak rock. Piles will probably refuse in moderately strong or stronger rock.

The evaluation of pile drivability for rock should be based on the RQD (Rock Quality Designation), the percent recovery, and the compressive strength of the rock. The RQD is defined as the percent ratio of the cumulative length of core samples 4-in. long or longer to the length of the core run. The percent recovery is defined as the percent ratio of the length of sample recovered to the length of core run. Very poor to poor quality rock is considered to have an RQD of less than 50 percent. Rock having a compressive strength of less than 100 ksi is considered to be weak.

During the installation of five platforms in the Zuluf Field, none of the piles could be driven through gypsum layers of 5 to 16-ft thickness. RQD ranged from 40 to 77 percent, and the unconfined compressive strength varied from 107 ksi to 301 ksi. The minimum percent recovery was 98 percent. At two other sites, only one pile of five at each site reached refusal. RQD was 0 and 47 percent, percent recovery was 83 and 70, but no unconfined tests could be run. The maximum blow counts observed for the remaining piles ranged from 32 to 391 blows per foot, with an average value of 160. None of the piles refused in gypsum layers at two locations where the RQD was 0 and 67, and the percent recovery was 52 to 80.

CONCLUSIONS

The following conclusions are based on the case histories presented in this paper. Although these conclusions are directly applicable to the Arabian Gulf, they may be used at other locations with some discretion.

1. For very dense sand, hard clay, or mixed soil profiles, a reasonable prediction of field blow counts is obtained using the lower and upper bound curves of soil resistance to driving for a plugged pile.

2. When computing the soil resistance to driving for a plugged pile, end bearing should not be limited to the frictional resistance developed by the soil plug.

3. For a sand profile, blow counts predicted using the static pile capacity or the resistance to driving computed for a coring pile will badly underestimate the field blow counts.

4. For a clay profile, the range in blow counts predicted assuming either a coring pile or a plugged pile may be fairly narrow. When the clay strata contain numerous seams and thin layers of rock, field blow counts will be erratic, and may exceed predictions for the upper bound plugged case.

5. For a mixed soil profile, blow counts predicted assuming either a coring pile or a plugged pile may be very similar.

6. Pile drivability for rock may be evaluated qualitatively. Refusal is likely when the RQD is greater than 50 percent, the percent recovery is greater than 85 percent, and the unconfined compressive strength of the rock is greater than 100 ksi.

ACKNOWLEDGEMENTS

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REFERENCES

Fig. 1 — Estimated resistance to driving — Case 1
Fig. 2 — Measured hammer performance data — Case 1

Fig. 3 — Comparison of observed and predicted blow counts — Case 1
Fig. 4 — Comparison of observed and predicted blow counts — Case 1
Fig. 5 — Comparison of observed and predicted blow counts — Case 2
Fig. 6 — Comparison of observed and predicted blow counts — Case 2
Fig. 7 — Estimated resistance to driving — Case 3
Fig. 8 — Measured hammer performance data — Case 3
Fig. 9 — Comparison of observed and predicted blow counts — Case 3
Fig. 10 — Comparison of observed and predicted blow counts — Case 3
Fig. 11 — Comparison of observed and predicted blow counts — Case 4
Fig. 12 — Estimated resistance to driving — Case 5

Penetration Below Seafloor, Feet

Coring - Lower Bound
Plugged - Lower Bound
Very Stiff to Hard Calcareous Silty Clay
Dense Carbonate Silty Sand
Medium Dense Carbonate Fine Sand
Firm to Very Stiff Calcareous Silty Clay
Medium Dense Carbonate Sand

Estimated Soil Resistance to Driving, Kips

0 500 1000 1500 2000 2500 3000

175 150 125 100 75 50 25 0
Fig. 13 — Comparison of observed and predicted blow counts — Case 5
Fig. 14 — Comparison of observed and predicted blow counts — Case 5
1. A survey of numerical methods in offshore piling

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A review of numerical methods used in the offshore piling industry is given. The purpose of these is to compute the static and dynamic behaviour of single piles and of pile groups at working load and at 'failure' for various types of loading, principally axial and lateral. Using discrete element methods (the 't-z' and 'p-y' types of calculation) it is possible to follow accurately the load transfer mechanism operating between single piles and the ground up to collapse, for monotonic and cyclic loading. The calculation is well within the scope of mini-computers, subject to adequate discretization of the ground's resistance. Back-analyses are essential. Using finite element methods, load transfer can be predicted, for single piles at working loads, using ground resistance parameters which are in everyday use in soil mechanics. It is within the scope of faster computers to continue such computations up to collapse, and this is a likely development. A particularly interesting case of 'failure' of single piles occurs during installation. Attempts to link driving resistance with ultimate static capacity still depend on the use of parameters which are not measured every day in soil mechanics laboratories. Drivability itself is reasonably predicted, particularly at soft sites, and back-analyses are encouraging. Flutter has been identified as an instability mechanism worthy of consideration. Analysis of pile group behaviour still rests heavily on the assumption of linear ground properties and on the principle of superposition. Errors of 20% or more are not uncommon in the computation of influence factors for single piles if the pile is inadequately represented. Effective stress analyses depend on a better knowledge of excess pore pressures due to driving, and on more realistic ground stresses after installation than can be measured or computed at present. Dynamic analyses of piles and groups in situ (as distinct from during driving) subject to wave and earthquake loading are at an early stage of development, but will clearly be pursued intensively in the future.

INTRODUCTION
Numerical methods have been widely used in the offshore piling industry for the past 25 years or so. This paper cannot attempt to be a literature review, but merely sets out the current state of achievement and tries to point the way to future developments. In addition it draws together the various strands of work presented at this Conference. These fall naturally into four main subject areas, namely, quasi-static behaviour of piles and groups, drivability, pore pressure considerations and dynamics.

QUASI-STATIC BEHAVIOUR OF PILES AND PILE GROUPS

2. Deflexion: the t-z method. A basic problem is the calculation of the deflexion of a single axially loaded pile subjected to prescribed load or displacement at the head. Even this problem is quite intractable without resort to numerical methods. For these purposes the pile can be discretized as a series of finite difference stations or as a series of linear 'finite elements', and the ground as a series of discrete axial 'springs' or as some continuously distributed axial spring stiffness (Fig. 1). Although called springs, the ground resistance-displacement relationships can be as complicated as is necessary. These are usually called t-z curves and can be introduced into the computer program either as mathematical functions or, more usually, as a series of points between which linear interpolation is assumed (Fig. 2).

3. Various methods can be used in the computation to follow the prescribed t-z curves. The most modern and efficient are borrowed from genuine finite element analysis and work with a constant stiffness in each ground spring (e.g., the slope of the first segment of the t-z plot (Fig. 3)). As loading proceeds, any excess force in a spring over and above the force which ought to be carried for that value of z is redistributed to the other springs by processes called 'initial stress' or 'viscous strain' in finite element work. In these methods, the simultaneous equations have constant coefficients and are merely solved for varying loads. Displacements at the pile head rather than forces should be prescribed for two reasons. First, it is more efficient since fewer iterations are required in the numerical process (typically two before failure is approached) and secondly, displacement control is the only way of continuing the analysis beyond peak load on the pile. Some of the older finite difference algorithms are rather cumbersome by modern standards.

4. This type of calculation is well within the scope of mini-computers. Of course, the difficulty lies in selecting the t-z curves appropriate to various soil types and conditions. The suggestion has been made of the dimensionless relationship

\[
\frac{t}{t_{\text{max}}} = 2 \left( \frac{z}{z_c} \right)^{\frac{15}{7}} - \frac{z}{z_c}
\]

for the side springs, where \( t_{max} \) is the maximum soil resistance which is mobilized at a critical displacement \( z_e \). For the end bearing spring, the corresponding suggestion is

\[
\frac{t}{t_{max}} = \left( \frac{z}{z_e} \right)^{1/3}
\]

5. Figure 4 shows how field data from test piles in sand and clay can be back-analysed using this approach. A large number of such fits would be necessary to build up confidence in the use of the method in new situations.

6. **Deflexion: finite element methods (Paper 11)**. An improved representation of the ground is as a solid, in the simplest case an elastic solid bonded to the pile. However, mesh design problems arise when modelling pipe piles with open or closed ends. It is difficult to achieve both the right end bearing area and the right pile stiffness at that diameter when analysing equivalent solid piles.

7. The real benefits come when non-linear, stress-dependent properties are taken for the soil, together with slippage allowance between pile shaft and soil. For example, Desai originally showed that the load transfer in a pile in sand is quite non-uniform with depth, as shown in commentary calculations in Fig. 5. When a field test was independently by this method, the load-displacement curve and load transfer profile were rather well reproduced in such calculations. However, the non-linear elastic assumption for the soil and interfaces means calculation becomes unreliable when a large number of elements 'fail', so ultimate loads are best not computed in this way.

8. **Deflexion: boundary element methods (Paper 14)**. Particularly when ground conditions are uniform and linear stress-strain properties can be assumed to prevail, boundary element methods can be superior to finite elements because fewer equations have to be solved. Poulos and Davis have provided widely used charts based on a simplified form of this method, subsequently somewhat refined by Butterfield and Banerjee. For layering and other forms of non-homogeneity, or when non-linear soil properties have to be considered, the method is less attractive.

9. **Failure**. The \( t-z \) computations for load-deflexion can be continued to collapse, and by means of displacement control can take residual conditions into account. The non-linear elastic type of finite element calculation is not recommended for computing collapse. Instead, initial stress or viscoplastic strain algorithms should be used. Examples of displacement fields at collapse of deep foundations in cohesive and cohesive-frictional materials are shown in Fig. 6, together with load-deflection graphs for base pressure. By these means, the bearing capacity factors \( N_c, N_q \) and \( N_p \) can be obtained numerically and the load transfer mechanism at failure identified.

10. **Boundary element methods** can of course in principle be used in this area, but have not so far found practical application.

11. **Cyclic loading (Paper 16)**. An important feature of offshore loading conditions is their cyclic nature. It is well known that under (slow) cyclic loading, engineering materials degrade and become softer and weaker. Because of their particulate nature, clays are prone to the formation of low strength, slickensided rupture surfaces under large and repeated alternating displacements. The \( t-z \) and finite element methods can cope with cyclic loading, given that the material behaviour can be defined.

12. For example, Fig. 7 shows a possible \( t-z \) behaviour for side springs under cyclic loading. Peak \( t \) is a function of \( N \), the number of cyclic load applications, as is the ratio of peak \( t \) to displacement \( z \) at which it is attained. Fig. 8 shows typical results of this kind of computation for varying cyclic load (displacement) amplitude. At lower levels stabilization takes place but as the level increases, the pile fails in cyclic loading. Tip resistance can be ignored in tension and so on.
Fig. 4. Back-analysis of field results by t-z method: (a) closed-ended pile in sand; (b) pile with shoe in clay

Fig. 5. Back-analysis of field results by finite element method: (a) load distribution of closed-ended pile; (b) graph of t with depth for various load increments

Fig. 6. Viscoplastic analysis of deep foundation in cohesive–frictional material
13. Another feature is the generation and dissipation of pore pressures during cycling. Finite element approaches can deal with this, but practical cases do not seem to have been solved yet.

Groups of axially loaded piles

14. Deflexion (Papers 14 and 15). For deflexion of groups of axially loaded piles, the boundary element methods come into their own. The $t-z$ approach ignores interaction completely, and the three-dimensional nature of the problem makes finite element computations expensive, even for linear soils. Therefore linearity of the soil's stress-strain response is usually assumed for the purposes of interaction computations, and so boundary elements are attractive. Charts have been produced by Poulos and by Butterfield and Banerjee which enable such interaction factors to be computed on a computer using the geometries. Sometimes the interaction factors from such a linear analysis are combined with $t-z$ curves for single piles to yield an empirical non-linear group behaviour.

![Fig. 7. Degradation of ground: (a) static loading; (b) cyclic loading](image)

15. Failure. Three-dimensional finite element computations can be done, but are rather expensive. If an equivalent axisymmetric pier can be assumed, stability parameters follow.

16. Cyclic loading. The writer is not aware of work of this nature.

Single laterally loaded pile

17. Deflexion: the $p-y$ method (Paper 17). The computations in the $p-y$ method are entirely analogous to those of the $t-z$ method, with $p$ replacing $t$ and $y$ replacing $z$. As long as satisfactory curves can be specified this calculation is very quick and cheap.

18. Deflexion: finite element methods (Paper 12). For axisymmetric pile piles the commonly used simplifications for semi-infinite structures under non-axisymmetric loads can be used. Displacements and so on are expanded in Fourier series so that the analysis becomes quasi-two-dimensional. As long as the geometry remains axisymmetric there is no difficulty in incorporating layered soils. Typical results for the deflexion of a pipe pile under lateral load are shown in Fig. 9, together with those originally published by Poulos, who assumed that the pile was a thin rectangular strip, and used a boundary element method. The latter can overestimate deflexions by 25% or more. The power of the finite element method is shown in Fig. 10, where deflexion profiles in layered soils are given. The radius of influence is computed to be typically 10 pile diameters for homogeneous soil and 6 pile diameters for soils increasing in stiffness with depth. The depth of influence is never greater than about 6 pile diameters. In the boundary element method (e.g., Banerjee and Davies), rather radical simplifications have to be made to cope effectively with arbitrary inhomogeneity. The finite element technique has been extended to consider non-linear soils. The difficulties here are merely in storing enough information about the circumferential variations in properties.

![Fig. 8. Pile analysis by $t-z$ method allowing for cyclic degradation: (a) load path of typical spring; (b) load-displacement response of pile](image)
19. **Deflection: boundary element methods (Paper 13).**

With the provisos given above, the boundary element technique is quite suitable for analysis of linear problems, especially in homogeneous soils.

20. **Failure.** Because of the very high bending moments at the mudline carried by piles when laterally loaded, material failure in the piling is much more likely than soil failure for deep-driven offshore piles. Stiff, straight piles could fail by a rotating mechanism but this does not seem to be of practical interest.

21. **Cyclic loading.** Comments are entirely analogous to those made about axial loading.

**Groups of laterally loaded piles (Papers 13 and 18).**

22. For groups of laterally loaded piles, the boundary element method has again seemed the natural choice. As was the case with axially loaded piles, empirical marriage of linear interaction factors with non-linear $p-y$ response is often attempted to obtain practical solutions.

**DRIVABILITY**

*The one-dimensional wave equation*

23. Because of the difficulties of ordering equipment ahead of time, fleeting weather windows and so on, predictions of pile drivability have assumed great importance in offshore operations. The techniques are not widely used in on-land situations, in the UK at least. The original recommendations of E. A. L. Smith with respect to material parameters such as elastic compressions appear to be adhered to. Probably the major difficulty attaches to the estimate of the viscous component of resistance (Smith’s parameter $J$). It has been pointed out that the success of the method in estimating pile set may well be due to the insensitivity of this quantity to the method of computation. Other factors, which are often measured in instrumented pile tests, are much more sensitive to the method of computation employed. It is also fair to say that predictive capacity has turned out to be much better in soft as opposed to hard sites, especially for clays and clay–sand mixtures.

**Special problems of offshore piling (Papers 4–7)**

24. Apart from the large scale of the operations, involving very massive piling and novel capacities of driving equipment, special problems concern, for example, gravity connectors. Offshore piles tend to be driven through a long follower at the end of which is a heavy connector. Thus in the wave propagation procedure there is a significant reflection back up the pile from the commutator, and moreover a separation (or bending, stress-transmission) occurs. Nevertheless, impressive back-analyses of driving records on some sites have been achieved. If this can be done, an obvious extension is to analyse the driving record in situ and hence to predict the ultimate static capacity of the pile, thus preventing costly over-driving. Considerable experience of these techniques has been built up on land sites in certain areas of the world and it remains to be seen how general the extrapolation procedure is. Again the difficulty in the drivability phase is the viscosity effect, and one would expect ‘sands’ to be more amenable to prediction than ‘clays’. In addition the phenomenon of set-up due to pore pressure effects is clearly an additional difficulty in ‘clays’.

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**SURVEY OF NUMERICAL METHODS IN OFFSHORE PILING**

**Effect of curvature and/or kinks**

25. Conductor piles can be deliberately driven with a curvature, the better to exploit the resources in a reservoir. Alternatively piles can be imperfectly welded so that there is an induced curvature or even a sharp kink between adjacent sections. Because of the great length of conductors particularly, concern has been expressed as to the effects of such disturbances on the drivability predictions, and on the stresses in the piling and forces on the guides.

26. The problem has been tackled in a rather mathematical way by Fischer, in the form of finite difference

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**Fig. 9. Finite element analysis of laterally loaded pile in uniform ground; $L/D = 25$**

**Fig. 10. Laterally loaded pile in layered ground; $L/D = 25$, $x = 0$**
approximation to a rather obscure pair of coupled differential equations formulated by Isakovitch and Komarova. A simpler method of attack, which shows immediately whether curvature and/or kinks have any great significance, involves finite element approximations of typical piles. As shown in Fig. 11, the pile elements can be genuinely curved, or curves can be approximated by a series of kinks. In either case there will obviously arise coupling between the compressional wave and the transverse motions of the pile. If the curved pile is a reasonable approximation to the truly curved one, this representation is clearly preferable, since arbitrary kinks can readily be treated.

27. If curvature is to have any effect on the driving process it must be manifested in a shift of the eigenvalues of the curved pile relative to those in the straight pile. Figure 12 shows the eigenvalues for three cases, namely straight, truly curved, and kinked piles. It can be seen that the differences between non-straight and straight piles are quite small, as are those between truly curved and kinked. On this basis one would expect the effects of curvature on drivability to be small for typical curvatures.

28. A second analysis involves the stresses in the piling (compressional plus flexural) during driving. Fig. 12 shows a representation of a pile which failed due to overstressing during driving. The computed stress in the pile at a point close to the failure position is shown in Fig. 13, from which it can be seen that the additional stress due to flexure was a second-order effect and could not really have contributed much to the failure. Material imperfection is a more likely cause.

Flatter (Papers 2 and 3)

29. Recently, attention has been drawn to the nature of the soil forces which resist the penetration of piles during installation. It has been pointed out that these forces may be of the 'follower' type (i.e., they remain tangential to the pile rather than taking up some fixed (usually vertical) direction). This being so, instabilities of a type frequently encountered in aerodynamics can be met at load levels far short of those required to cause instability in the classical buckling sense. Fig. 14 shows typical results in terms of load combinations at which various types of instability occur. The finite element method proves to be a particularly simple means of solving these problems. So far, publications have merely indicated the possibility of instability arising; they have not shown the effects on drivability of a tendency towards instability. This tendency again manifests itself as a shift in the eigenvalues of the pile—soil system and can readily be incorporated in drivability programs.

Three-dimensional effects

30. In hard driving, it seems quite likely that significant energy is expended in deforming the pile laterally against the sides of the hole. In addition, the presence or absence of a soil 'plug' inside the pipe can clearly affect the mechanisms of wave transmission. It is perfectly possible to analyze the influence of these factors using axisymmetric finite element representations of pile and soil in a dynamic simulation of this sort, and it seem to have been achieved.

PORE PRESSURE EFFECTS (Papers 19 and 20)

31. So far, soil resistance and strength have been represented exclusively in terms of total stresses. However, it is well known that, particularly for piles driven into normally consolidated, impermeable clays, large excess pore water pressures can be generated, the dissipation of which governs the pile's ability to resist loads applied at various times after driving.

32. Numerical methods have recently been applied in this area, but the problem is a difficult one, and the writer doubts whether the state of excess pore pressure existing adjacent to piles driven into normally consolidated or overconsolidated clays and sand—clay mixtures, such as those in the North Sea, can be predicted with much confidence analytically. Field observation seems to be necessary here. However, rates of dissipation of the generated excess pore pressures should be perfectly adequately computed by present analytical techniques, given adequate values of permeability coefficients, obtained from tests in the field or on large specimens.

DYNAMICS (Papers 8—10)

33. In a cyclic loading environment the frequencies of the alternating forces and their relationship to the critical frequencies of the structure—soil system assume a decisive importance. The main types of dynamic loading experienced by offshore structures appear in the form of sea waves and/

<table>
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or earthquakes. These two forms of excitation are radically different, as has been pointed out in the context of gravity structures. For piled structures, many of the same considerations apply.

**Sea waves and earthquakes**

34. Ocean wave loading is of low frequency and long duration, whereas earthquake loading is of high frequency and short duration. The magnitudes of total load imposed on the structure-soil system by the two excitations are, however, of the same order. In the case of excitation by earthquakes, it is usually assumed that shear waves propagate vertically through the soil from bedrock, and of course all of the soil and any deep-driven piles embedded in it are subject to the earthquake motions. These lead to rather large shear strains everywhere in the soil mass and a great deal of 'primary' interaction between the soil and the piles. This interaction causes shaking of the structure which in turn exerts a 'secondary' interaction back through the piling into the soil. The descriptions primary and secondary dynamic interaction are borrowed from usage in the nuclear power plant industry and may not realistically express the relative importance of the two effects. When the excitation is by sea waves, the interaction is by the above definition totally secondary. This type of interaction is much more localized to the immediate vicinity of the structure, and in the case of typical deep-piled offshore structures, will not be felt at all by the ground below quite a shallow depth of a few metres below mudline.

**Methods of analysis**

35. Idealizations of piled foundations for dynamic analysis follow closely their static counterparts. Often the

![Fig. 11. Representation of curved conductors by finite elements: (a) curved pile; (b) curved elements; (c) straight elements](image)

![Fig. 12. Idealization of failed pile](image)

![Fig. 13. Stress computations for failed pile](image)

![Fig. 14. Flutter analysis of piles by finite elements: F = total force on pile, P = total pile length, \( \eta = kl^4 / EI \)](image)
ground resistance is approximated by a set of springs, but this method is more questionable than in the static case because of the inability of the springs to cope effectively with a major source of energy dissipation, namely by geometric or radiation damping. This is particularly important in earthquakes where there is a lot of energy present throughout the excited ground. In order to include a measure of geometric damping, finite element or boundary element approximations can again be used, and a few solutions for simple cases can even be produced analytically.

Among the various computational tactics which can be adopted, the most widely used are integration of the equations of motion in the time domain, which allows other general energy dissipation mechanisms to be included. These tend to comprise the non-linear effects of friction, plasticity, viscosity and so on. Alternatively, linearised calculations can be made by the 'response' or 'impedance' methods, taking account of hysteretic damping only. Both methods have their advantages, and experience with the analysis of gravity platforms indicates that lateral stiffnesses and motions of the structure-soil system will probably be similarly predicted by both methods, given similar initial assumptions. However, the truly non-linear calculations produce some results, such as permanent displacements and sub-resonances, which cannot be present in any linearized analysis. This is a fruitful field for further study.

CONCLUSIONS

Numerical methods in offshore piling are in many cases more sophisticated than the physical data which is input to the programs. What is often required is back-analysis from the field since full-scale tests are prohibitively expensive. Modelling, especially true scale modelling using centrifuges, can also contribute valuable data for calibration of the numerical results.

Areas where further numerical developments can be made include constitutive relationships for soil, especially during cyclic loading and penetrating of piles, threedimensional effects and dynamics.

REFERENCES

15. GOBLE G. G. et al. Bearing capacity of piles from dynamic measurements. Case Western Reserve University, Cleveland, 1975, final report to Ohio Dept of Transportation.
THREE-DIMENSIONAL ANALYSIS OF PILE DRIVABILITY

I.M. Smith, University of Manchester, England
Y.K. Chow, Fugro Ltd., Hemel Hempstead, England

SUMMARY

Pile drivability is usually assessed, in the offshore industry as well as elsewhere, on the basis of calculations which solve the one-dimensional wave equation. Clearly this is an approximation to the real situation, in which the driving process induces stress waves in the soil surrounding the pile. This paper examines the validity of the one-dimensional approximation, by comparing it with a three-dimensional (axisymmetric) one. Both one- and three-dimensional idealisations are of the finite element type.

In addition to the long-established use of wave equation predictions for drivability, recent advances in pile instrumentation during driving have led to the use of the one-dimensional equation as a means of analysing driving records with a view to predicting static capacity. In this sort of calculation, a few passages of the stress wave along the pile are analysed. In this paper, one-dimensional and three-dimensional models are compared in this context also.

Finally, offshore piles are usually driven as open-ended hollow pipes and controversy exists as to how to treat the soil "plug" in one-dimensional analyses. The ability of the three-dimensional analyses to shed light on this problem is briefly discussed.

THE ONE-DIMENSIONAL WAVE EQUATION

The classical numerical solution of this equation, proposed thirty or more years ago by E.A.L. Smith, still forms the basis of modern computer programs. The discretisation of pile and soil is illustrated in Figure 1 (right) and is based on a finite difference approximation to the governing differential equation. The mass and stiffness of the system are traditionally discretised as shown, the mass of the pile
Fig. 1 Method of Representing Pile

Fig. 2 Load-Deformation Soil Spring
being "lumped" and the mass of the soil surrounding the pile ignored. That is, no attempt is made, as in the analysis of machine foundation vibrations for example, to consider any soil to act as an "added" mass to the vibrating system. An exception to this rule, suggested by Hereema and de Jong for the driving of open-ended pipe piles offshore, involves the addition of masses and springs to represent a soil "plug" inside the hollow pile.

The soil resistance, $R$, is traditionally assumed to be elasto-viscoplastic as illustrated in Figure 2. The ultimate static capacity, $R_u$, of each soil "spring" is estimated independently and the elastic displacement or quake, $Q_u$, is still often taken to be the 2.54mm (0.1 inch) originally recommended by E.A.L. Smith. Thus in the absence of viscosity, a soil spring would traverse load path 1 in Figure 2.

In order to obtain a better fit with field data, soil springs are actually assumed to follow something like path 2 in Figure 2. The added resistance is ascribed to velocity-dependent viscous effects but may actually be an attempt to cater for some radiation damping as well, as will be discussed later. At any rate, various viscous models have been proposed, in which the viscous resistance is proportional to velocity of penetration, static resistance, pile impedance, velocity of penetration to the power 0.2 and so on. These various viscous models are not the concern of the present paper. Table 1 shows some typical soil data used in offshore studies, and what becomes immediately apparent is the need in the calculations for the non-standard soil mechanics parameters $Q_u$, $J_s$, and $J_p$, the elastic quake, shaft damping and point damping respectively.

Figure 1 (left) shows a slightly different one-dimensional pile discretisation. This is based on a one-dimensional finite element model of the pile which permits the pile's mass either to be "lumped" as in the finite difference case, or continuously distributed along the element as it must be in reality. However, despite minor variations (for example the capblock is usually assumed to be massless in the traditional programs) the finite element and finite difference idealisations should give essentially the same results for the same physical assumptions. The authors prefer the finite element model for personal reasons of program adaptability, eg. Smith.
<table>
<thead>
<tr>
<th>Author</th>
<th>Field</th>
<th>Soil Type</th>
<th>Quake (mm)</th>
<th>Damping (s/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Side Tip</td>
<td>J_s</td>
</tr>
<tr>
<td>1. Naughton &amp;</td>
<td>Hondo Platform</td>
<td>Stiff OC clay</td>
<td>2.54</td>
<td>0.164</td>
</tr>
<tr>
<td>Miller (1978)</td>
<td></td>
<td></td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td>2. Young et</td>
<td>Thistle Platform</td>
<td>Stiff OC clay</td>
<td>2.54</td>
<td>0.49</td>
</tr>
<tr>
<td>al (1978)</td>
<td></td>
<td></td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td>3. Sutton et</td>
<td>Forties Platform</td>
<td>Lightly OC clays with layers of</td>
<td>2.54</td>
<td>0.656</td>
</tr>
<tr>
<td>al (1979)</td>
<td></td>
<td>dense sand</td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td>4. Sullivan &amp;</td>
<td>(a) Gulf of Mexico</td>
<td>V. soft to stiff clay</td>
<td>2.54</td>
<td>0.492</td>
</tr>
<tr>
<td>Ehlers (1972)</td>
<td></td>
<td>Soft to firm clay</td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Off Louisiana</td>
<td>Firm to v. stiff clay</td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td></td>
<td>coast</td>
<td></td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c) Off coast of</td>
<td></td>
<td>2.54</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Nigeria</td>
<td></td>
<td>2.54</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 1 Parameters Used in Offshore Drivability Predictions**

**ALGORITHM SENSITIVITY**

All numerical solutions of differential equations are approximate, the only question being the magnitudes of the errors committed. Figures 3 and 4 illustrate one aspect of algorithm sensitivity. Three well known algorithms were used to solve a one-dimensional problem (based on

---

**Fig. 3** Displacement-Time for Different Integration Schemes ($\Delta t = 5 \times 10^{-4}$)
the original example chosen by Smith\textsuperscript{1} with a fixed timestep. The algorithms were the conventional explicit method, a central difference implicit method labelled "Newmark" and a second implicit method popular in structural dynamics and labelled "Wilson". Figure 3 shows that the displacement of the pile tip, the parameter traditionally of interest in these computations, is insensitive to algorithm choice. However, the accelerations in the pile, see Figure 4, are quite different for the three different methods of computation.

A second aspect of algorithm sensitivity is shown in Figures 5 and 6.

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**Fig. 4** Acceleration–Time for Different Integration Schemes ($\Delta t = 5 \times 10^{-4}$)

---

**Fig. 5** Displacement–Time Plots (Wilson) (Consistent Mass)
where the test problem was solved by a typical algorithm (Wilson's actually) for a fixed timestep but varying space step sizes. Again pile displacement is seen to be insensitive to space discretisation while acceleration is highly sensitive.

The consequence of these sensitivity studies is that although an algorithm may work quite well in the role for which it was originally intended, care is necessary in new applications, Smith⁴. For example, these sorts of algorithm have traditionally only been used to calculate the "set" or plastic displacement of the pile tip, shown above to be an insensitive parameter. Indeed a survey by Saxena⁵ showed that most popular programs cease computing after the tip displacement reaches its first maximum value, and the set is inferred by subtracting the quake from that maximum. However, there is an increasing tendency to use one-dimensional calculations in conjunction with field monitoring of strains and accelerations in driven piles, eg. Goble and Rausche⁶. These computations usually need to run for a much longer time than that necessary to establish pile set, and clearly the scope for algorithmic errors is greater. Of course appropriate reductions of space and time steps should lead to a converged result, appropriate to that one-dimensional model.

Fig.6 Acceleration-Time Plots (Wilson) (Consistent Mass)
THREE-DIMENSIONAL ANALYSIS

For piles with axial symmetry, for example the pipe piles driven offshore, it is perfectly feasible to carry out computations of drivability using a three-dimensional axisymmetric model of the pile and its surrounding soil. Numerically this could be done in various ways, but the authors again employ the finite element method, a typical model of a (solid) axisymmetric pile being shown in Figure 7. Pile and soil are represented by 8-noded elements of quadrilateral cross-section as shown and thin 6-noded interface elements separate pile and soil to allow for relative slippage.

Such a model represents much more faithfully the realities of pile driving in that stress waves are generated in the soil mass. In order that these waves are not reflected from mesh boundaries, viscous dashpots forming a "standard viscous boundary" (Lysmer and Kuhlemeyer\textsuperscript{7}) are placed around the boundaries, Chow\textsuperscript{8}.

An advantage of this type of model is that more standard soil mechanics parameters can be used to describe the properties of the soil mass. The analyses described in this paper are restricted to piles in

Fig.7 Finite Element Model for Pile Drivability
"clay" which is assumed to be essentially undrained during driving. The clay is idealised as elastic–perfectly plastic yielding according to the von Mises criterion, and undergoing associated (no–volume–change) flow. The elastic properties are a Young's modulus, $E$, and a Poisson's ratio, $\nu$. The yield stress is a simple function of $C_u$, the undrained strength as measured in a triaxial or plane strain test.

Radiation damping now occurs in the model automatically, but in some analyses viscous damping has been included as well. When this is done, it applies only to the nodal points along the shaft and tip of the pile, in a manner similar to that used in the one-dimensional analyses.

To avoid any difficulties caused by plug movement, closed-ended pipe piles are considered first. For such piles the static tip resistance, $N_c$, in the formula $q_{ult} = C_u N_c$, is computed to be about 9, in line with current practice. The strength of the interface elements was taken to be $0.5 C_u$ to take some account of remoulding during driving.

The Wilson implicit algorithm was used to integrate the three-dimensional wave equation and in some calculations, the initial conditions were taken to be a prescribed ram velocity. However, another method for offshore piles is to use the manufacturer's force–time plot for a given hammer and pile combination, and this has been used in some of the calculations also. (These initial conditions have previously led to difficulties in analysing diesel–driven piles using some standard programs). Nonlinearity is accounted for by the "initial stress" type of algorithm.

COMPARISON OF THREE–DIMENSIONAL AND ONE–DIMENSIONAL MODELS

(a) Wave propagation down a pile in the absence of soil resistance.

Case 1: End bearing pile.

A simple test case was selected for which an exact solution is known, namely a uniform rod, fixed at one end and free at the other and subjected to a constant impact force. Figure 8 shows a comparison of one-dimensional finite element solutions with the true solution for both lumped and consistent mass assumptions. The latter are clearly superior although the contrary is sometimes claimed in the literature.

The corresponding comparison for the three-dimensional finite element approximation is shown in Figure 9. Here, all the numerical solutions adopted the consistent mass idealisation and it can be seen
Fig. 8 Axial Wave in 1-D Rod (1-D Elements)

Fig. 9 Axial Wave in 3-D Rod (Axisymmetric Elements)
that Poisson's ratio has a noticeable effect on stresses although not on axial displacements. However the best comparison should be when Poisson's ratio is zero and the quality of the three-dimensional numerical solutions is very similar to that of the one-dimensional ones.

Case 2: Free-free pile (no soil resistance).

The pile data are given in Table 2. The initial conditions were a prescribed hammer velocity and Figure 10 shows a comparison of the pile tip displacements.

<table>
<thead>
<tr>
<th>Hammer Data</th>
<th>MRSS 750</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>75 kN</td>
</tr>
<tr>
<td>Impact Energy</td>
<td>84.4 kNm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Capblock</th>
<th>Diameter = 0.72m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>0.18m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Anvil</th>
<th>Weight = 31.5 kN</th>
</tr>
</thead>
</table>

| Pile         | Outer Diameter = 1.524m |
|--------------| Wall Thickness = 63.5mm |
|              | Total Length = 36m      |

<table>
<thead>
<tr>
<th>Soil</th>
<th>Pile Penetration = 20m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cohesive</td>
</tr>
</tbody>
</table>

**TABLE 2 Driving Accessories and Pile Data**

![Displacement-Time for Pile Tip (No Soil Resistance)](image-url)

Fig.10 Displacement-Time for Pile Tip (No Soil Resistance)
The agreement is very close, and the same can be said of velocities and accelerations. In this case, Poisson's ratio had no marked effect on the three-dimensional solutions for axial displacement etc. as before.

(b) Wave propagation down a pile embedded in "clay".

The pile data are again as listed in Table 2, but the pile is now embedded to a depth of 20m in an elasto-plastic idealisation of undrained clay.

Case 1: Soil with $C_u = 100 \, \text{kN/m}^2$, $E = 1000 \, C_u$, $\alpha = 0.5$, $\rho = 0.01 \, \text{t/m}^3$. No viscous damping.

This example is intended to demonstrate the difference between the one-dimensional and three-dimensional analyses due solely to the stiffness of the soil surrounding the pile. In the one-dimensional case the soil spring stiffness is defined by the traditional quake value of 2.54mm, while in the three-dimensional model the soil's mass is assigned a very low value. No viscous damping is taken into account. Figure 11 shows the comparison of the computed tip displacements. They are clearly quite similar in terms of hills and hollows but the curves are displaced relative to one another. This can be ascribed to small inertia effects in the three-dimensional model together with the differences in soil stiffness and distribution.

When the soil strength and the Poisson's ratio of the pile were varied, essentially similar results to Figure 11 were obtained.

![Displacement-Time for Pile Tip](image)

Fig. 11 Displacement-Time for Pile Tip ($\rho_{\text{soil}} = 0.1$, $C_u = 100$)
Case 2: Soil with $C_u = 100$ kN/m$^2$, $E = 1000$ $C_u$, $\alpha = 0.5$, $\rho = 2.1$ t/m$^3$. No viscous damping.

In this case the influence of soil inertia was fully taken into account in the three-dimensional model. Figure 12 shows the comparison of the tip displacements which are quite close at small times after impact, but progressively diverge with time, the three-dimensional model giving the more damped response leading to an equilibrium "set" of some 4mm. The one-dimensional calculation did not achieve equilibrium although if the E.A.L. Smith suggestion of subtracting 2.54mm from the first peak displacement is used, a rather similar set of some 5mm would be predicted. Of course it could be argued that the viscous parameter $J$ in the one-dimensional analysis is partially intended to model some radiation damping. Its inclusion reduces the one-dimensional set by about 1mm for this clay strength.

When the soil strength is raised to 500 kN/m$^2$, equivalent to the hardest clays encountered in the North Sea, Figure 13 shows the comparative tip displacement plots. The curves diverge even earlier for the stronger soil with the three-dimensional calculation being very strongly damped, leading to an equilibrium set of some 0.7mm. When the Smith method is applied to the one-dimensional result, a quite similar set of some 0.5mm is predicted. However, if viscous damping is now included in the one-dimensional model, a zero set would be predicted, i.e. the pile could not be driven.

Results for weak clay, $C_u = 25$ kN/m$^2$, give the closest agreement

---

Fig. 12 Displacement-Time for Pile Tip ($C_u = 100$ kN/m$^2$)
between one-dimensional and three-dimensional models. A noteworthy observation is that in this case the first peak on the displacement-time curve is not the maximum, which is achieved later on.

Thus from these studies it can be inferred that the one-dimensional calculation can accurately predict pile set, but that in detail the performance of the pile is not closely followed, especially for stiff soils.

Additional computations using double and treble the hammer energy led to essentially the same conclusions.

Fig. 13 Displacement-Time for Pile Tip \( (C_u = 500 \text{ kN/m}^2) \)

Fig. 14 Pile Drivability: Two-Layer Problem
PARAMETRIC STUDIES USING THE THREE-DIMENSIONAL MODEL

The basic geometry is shown in Figure 14, the pile details being again those of Table 2. In the studies, the properties of layer 1 were kept constant while those of layer 2 were varied.

Figure 15 shows the effect of increasing the undrained strength of the clay in layer 2, keeping the other soil parameters constant. The effect is to decrease the maximum tip displacement and to increase the mobilised end resistance as one would expect.

A second point worthy of study is the influence of the clay's elastic stiffness on the results for a constant clay strength. Much evidence for stiffer clays suggests that in the equation $E = \beta C_u$, the factor $\beta$ is of the order of 500 - 1000. With the undrained strength of layer 2 kept at 100 kN/m², $\beta$ was varied from 250 to 1250. Defining the end bearing resistance factor during driving as $N_D$, where

$$N_D = \frac{\text{Maximum Tip Load}}{\text{Pile Tip Area} \times \text{Undrained Clay Strength}}$$

it can be seen from Figure 16 that $N_D$ is usually considerably higher than the $N_c$ of about 9 computed and measured in static analyses and tests. For $\beta$ values in the range 500 to 1000, $N_D$ increases from about

Fig. 15 Variation of Maximum Tip Response with $C_u$ (Constant $E$)  
Fig. 16 Sensitivity of Parameters to $\beta(E = \beta C_u$, Constant $C_u$)
16 to 20.5, and this must be attributed to inertia and radiation damping effects alone, since no clay viscosity was assumed.

Mobilisation of \( N_D \) During Driving

The above results suggest that \( N_D \) varies with \( C_u \) and with \( \beta \), and Figure 17 shows further evidence. In these results \( C_u \) for layer 1 is 100 kN/m\(^2\). The same results, for the case \( \beta = 1000 \), are plotted in Figure 18 in terms of mobilised tip displacement. The effect of changing \( C_u \) in layer 1 to 25 kN/m\(^2\), other factors remaining constant, is shown in Figure 19. Somewhat higher \( N_D \) values result.

The strongest trend apparent is the decrease of \( N_D \) with increasing soil strength. Confirmation of this comes from cone penetration tests in the North Sea conducted by Fugro, see Figure 20. This shows \( N_D \) values as high as 10 for some weak clays, reducing to less than 10 for some of the strongest clays.

Quake Values

Figures 18 and 19 allow an estimate of the tip quake computed in the three-dimensional analyses. If this is identified roughly as the tip displacement when \( N_D \) reaches its maximum, it can be seen to be

---

**Fig. 17** Variation of \( N_D \) with \( C_u \)

**Fig. 18** \( N_D \) During Driving (Closed-Ended Pile)
remarkably insensitive to soil strength and elasticity. Indeed the full range is only from about 2mm to 4mm, the higher values being for the softer soils. For typical clays encountered offshore, 2.5mm is a remarkably good estimate, in agreement with E.A.L. Smith's original suggestion.

Fig. 19 $N_d$ During Driving (Closed-Ended Pile)

Fig. 20 Variation of Cone Resistance with $C_u$
Estimation of shaft quake is more subjective. However Table 3 shows the mobilised maximum shaft force for a pile driven into uniform soils with varying shear strengths. If the side quake is identified with the pile head amplitude during driving, again a value remarkably close to 2.5mm is computed.

<table>
<thead>
<tr>
<th>Undrained Shear Strength(kN/m²)</th>
<th>Max. Mobilised Shaft Force(kN)</th>
<th>Proportion of Shaft Resistance</th>
<th>Pile Head Movement(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>1191</td>
<td>1.00</td>
<td>2.66</td>
</tr>
<tr>
<td>50</td>
<td>2377</td>
<td>0.99</td>
<td>2.77</td>
</tr>
<tr>
<td>100</td>
<td>4651</td>
<td>0.97</td>
<td>2.66</td>
</tr>
<tr>
<td>200</td>
<td>9237</td>
<td>0.96</td>
<td>2.63</td>
</tr>
<tr>
<td>300</td>
<td>13140</td>
<td>0.91</td>
<td>2.61</td>
</tr>
<tr>
<td>400</td>
<td>16570</td>
<td>0.87</td>
<td>2.74</td>
</tr>
<tr>
<td>500</td>
<td>19390</td>
<td>0.81</td>
<td>2.73</td>
</tr>
</tbody>
</table>

**TABLE 3** Mobilised Skin Friction with Movement of Pile Head

**Use of Driving Records to Estimate Static Capacity**

A typical approach, e.g. Gravare et al.\(^{10}\), predicts the total pile resistance (static capacity) according to the formula

\[ R_{\text{total}} = \frac{1}{2} \left( \text{Sum of pile top forces at times } t_1 \text{ and } t_2 \text{ derived from strain measurements} \right) + \frac{1}{2} \left( \text{Difference of pile top forces at times } t_1 \text{ and } t_2 \text{ derived from velocity measurements} \right) \]

The times \( t_1 \) and \( t_2 \) are taken to be \( 2L/c \) apart, where \( L \) is the pile length and \( c \) the speed of the wave propagation. Time \( t_1 \) is that at which the pile top force first reaches a maximum. Typical results obtained using this formula for the pile described previously, embedded in clays of strength 25, 100 and 500 kN/m², are shown below:

<table>
<thead>
<tr>
<th>( C_u )</th>
<th>25</th>
<th>100</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computed</td>
<td>2,500</td>
<td>8,500</td>
<td>20,000</td>
</tr>
<tr>
<td>Actual</td>
<td>1,607</td>
<td>6,430</td>
<td>32,148</td>
</tr>
</tbody>
</table>

In order to obtain these results the peak forces in the vicinity of \( t_2 \) were taken, although they occurred at different times in the two models. The agreement with static capacity is only of a general nature. Skill is therefore necessary in the prediction of pile capacities in
this way.

**Plugging of Pipe Piles**

By modelling open-ended piles using axisymmetric finite elements, it is possible to take account of the soil plug and soil-pile interface inside a hollow pile. Lack of space precludes a detailed treatment herein, but the finite element results show that the plug behaves quite differently in static and impact circumstances. In the former, skin friction is not fully mobilised on the inner wall of the pile and so the statically loaded pile behaves as "plugged". However, in the impact driving case, the inertia of the plug comes into play, and full skin friction tends to be mobilised inside the pile so that during driving the pile behaves as "unplugged". This may help to explain the observation that closed-ended piles can be easier to drive in clays than open-ended ones.

**CONCLUSIONS**

One-dimensional and three-dimensional models of impact driven piles have been compared. There are significant differences in behaviour, particularly for piles driven in stiffer, stronger soils.

**REFERENCES**


Impact and Longitudinal Wave Transmission

E. A. S. Smith, New York, N. Y.

The necessary formulas for a numerical method of calculation are derived without the use of calculus or other refined mathematics; an illustrative problem is solved in complete detail; eight different types of impact are discussed; methods are given for taking account of friction, humping, coefficient of restitution, plastic flow, and any branched systems are discussed briefly; methods for checking the calculated results.

Introduction

The purpose of this paper is to present, as simply as possible, an illustrated method that may be used with slide rule, desk calculator, or even a pocket calculator for approximate calculations such as the motion of a falling hammer, a weight striking a hammer, and the like. The method is arranged to provide long computation without mathematical error to any degree of precision, weight distribution, or other complications are involved; solving for such bodies has been discussed, and so forth.

Notation

The letters in parentheses designate particular angles and associated springs, and may be used as subscripts. The subscripts 1, 2, etc., are numbers designating time-intervals when used similarly to designate the initial instant.

The following apply to any time interval \( \Delta t \):

- \( f_1 \) displacement of weight measured from its initial position, m.
- \( f_2 \) compression of spring \( m \) m.
- \( f_3 \) force exerted by spring \( m \) on body.
- \( f_4 \) net force acting on weight \( m \).
- \( f_5 \) total weight of weight \( m \), lbf.

The following apply to the preceding time interval \( \Delta t \):

- \( f_6 \) displacement of weight \( m \) measured from its initial position, m.
- \( f_7 \) compression of spring \( m \) m.
- \( f_8 \) velocity of weight \( m \), fps.
- \( n \) spring constant for spring \( m \), lb/m.
- \( f_9 \) total force or resistance acting on weight \( m \), lbf.
- \( f_{10} \) acceleration due to gravity, 32.17 lbf/ps.
- \( f_{11} \) Young's modulus of elasticity, psi.
- \( f_{12} \) cross-sectional area, sq in.
- \( f_{13} \) length, m.
- \( f_{14} \) time interval used for numerical calculation, see.

The actual solution requires the following steps:

1. Set up a table with columns for the following:
   - \( f_1 \) for each interval \( \Delta t \).
   - \( f_2 \) for each interval \( \Delta t \).
   - \( f_3 \) for each interval \( \Delta t \).
   - \( f_4 \) for each interval \( \Delta t \).
   - \( f_5 \) for each interval \( \Delta t \).

2. Calculate the following:
   - \( f_6 \) for each interval \( \Delta t \).
   - \( f_7 \) for each interval \( \Delta t \).
   - \( f_8 \) for each interval \( \Delta t \).
   - \( f_{10} \) for each interval \( \Delta t \).
   - \( f_{12} \) for each interval \( \Delta t \).

3. Sum the results for each column for each interval \( \Delta t \).

4. Use the results to determine the final position of the weight after each interval \( \Delta t \).

Development of Basic Formulas

The desired quantity is the increment of displacement required during a single interval \( \Delta t \). This increment may be evaluated as:

\[ \Delta f = \frac{f_4}{f_{10}} \cdot \Delta t \]

The coefficient \( f_{10} \) is required because \( f_4 \) is expressed in inches and \( f_4 \) in feet per second. The required formula for \( f_4 \) is then:

\[ f_4 = f_4 \cdot f_{10} \cdot 12 \Delta t \]

The equation \( 1 \) is always known, because \( f_4 \) is the chosen time interval, and \( f_4 \) and \( f_4 \) are either given in the previous.

To obtain a formula for \( f_4 \), let the dotted squares in Fig. 2 represent the initial positions of weights \( m \) and \( m + 1 \), and let the solid squares represent their positions in interval \( \Delta t \). Then \( f_4 \) will be the initial length of spring \( m \) and \( f_4 \) will be its length in interval \( \Delta t \); also \( f_{10} \) and \( f_{10} \) will be the displacement of weights

Fig. 2
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<th>n</th>
<th>D₁</th>
<th>C₁</th>
<th>F₁</th>
<th>Z₁</th>
<th>V₁</th>
<th>D₂</th>
<th>C₂</th>
<th>F₂</th>
<th>Z₂</th>
<th>V₂</th>
<th>D₃</th>
<th>C₃</th>
<th>F₃</th>
<th>V₃</th>
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<td>−F₁</td>
<td>v₁+Z₁</td>
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</table>
SMITH—IMPACT AND LONGITUDINAL WAVE TRANSMISSION

4. Similarly, \( R_2 \) and \( R_3 \) are not only to oppose motion; therefore when \( F_1 \) becomes negative on line 12, \( R_2 \) becomes positive starting with line 13. Similar reversals of sign occur on lines 14, 20, 21, 22, and 23.

5. Spring \( K_3 \) is not designed to take tension; therefore, when \( F_1 \) becomes negative on line 21, it means that \( K_3 \) has ceased to act and \( W_1 \) has bounced off with a velocity of 7.5393 fps.

6. Impact ends in interval 24; therefore the duration of impact is approximately \( W_1 \) sec or 0.008 sec.

7. The numerical calculation may be started at the beginning of impact or at any other instant for which conditions are given. For instance, Table 2 might have been started at line 6 if the given conditions had been as follows:

\[
\begin{array}{ccc}
D_1 &=& 0.19722 \\
D_2 &=& 0.10295 \\
D_3 &=& 0.01975 \\
V_1 &=& 5.0156 \\
V_2 &=& 6.5367 \\
V_3 &=& 1.6665
\end{array}
\]

8. The numerical calculation may start from known forces. For instance, if the values of \( F_1 \) shown in Table 2 had all been

---

**ILLUSTRATIVE PROBLEM**

A simple problem shows how the foregoing formulas may be applied. A weight weighing 5 lb and moving at 10 fps strikes a stationary weight, each with a frictional resistance and elastic springs, as shown in Fig. 3. Calculations for motions and forces using a time interval of

---

**TABLE 1**

<table>
<thead>
<tr>
<th>Value of ( i )</th>
<th>( K_m ) lb</th>
<th>( R_m ) lb</th>
<th>( \Delta t ) sec</th>
<th>( \Delta \theta / W_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>100</td>
<td>9</td>
<td>0.004</td>
<td>1/400 8.3</td>
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<td>100</td>
<td>0</td>
<td>0.004</td>
<td>1/93 25</td>
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<td>0.05</td>
<td>160</td>
<td>0</td>
<td>0.004</td>
<td>1/186 3.5</td>
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</table>

Calculation may then be performed and tabulated in Table 2.

---

**TABLE 2**

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>( D_m )</th>
<th>( V_m )</th>
<th>( F_m )</th>
</tr>
</thead>
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<td>0.05</td>
<td>100</td>
<td>5.0156</td>
<td>0.19722</td>
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<td>0.05</td>
<td>160</td>
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---

*Fig. 3*

---

*Fig. 4*

**SOURCES OFF A**

---

**Fig. 4** PLOTTED VALUES FROM TABLE 2
It is assumed or assumed, the action of $W_2$, $K_2$, $W_3$, and $K_3$ could have been calculated exactly as in Table 2 except that columns $D_a$, $C_a$, $Z_a$, and $W_a$ could be omitted.

The resulting diagram for purposes of computation is shown in Fig. 8.

If the object is not continuous but consists of individual sections of considerable weight with intermediate springs, such as a railroad train, it ordinarily may be represented with sufficient accuracy as shown in Fig. 9 in which $W_i$ is the weight of the locomotive, and $W_2$, etc., are the weights of the individual cars. Any elasticity in the structure of the car itself may be added to the spring constant of the couplers at either end using Formula [8] given later. If extreme accuracy is required, each weight as noted may be divided into two or more weights with intermediate springs by following the method of Figs. 5 and 6.

Spring Constant $K_a$. If spring $K_a$ represents the elasticity of a unit length $l$ of a rod of uniform section, the formula for $K_a$ is

$$K_a = AE/l$$

If $K_a$ consists of two or more springs in parallel its value is obtained by adding the spring constants of the individual springs, thus

$$K_a = k_1 + k_2 + k_3$$

If $K_a$ consists of two or more springs in series its value is obtained by adding reciprocals, thus

$$1/K_a = 1/k_1 + 1/k_2 + 1/k_3$$

If $K_a$ represents a uniformly tapered square section with sides at one end equal to $a$ and at the other end equal to $b$, then

$$K_a = E\alpha^2 l$$

If $K_a$ represents a uniformly tapered round section of diameter $a'$ at one end and $b'$ at the other, then

$$K_a = \frac{\pi}{4} E\alpha^3 l$$

If $K_a$ represents a uniformly tapered rectangular section having sides $a''$ and $b''$ at one end and corresponding sides $a'$ and $b'$ at the other end, then

$$1/K_a = 1/a'b'' K_a = E\alpha^2 l$$

Stability. Stability may be defined as freedom from oscillation (for hunting) caused by a fault in the calculation and not present in the objects being analyzed.

One way to detect instability is to plot the values of $D_a$, $W_a$, and $K_a$ as in Fig. 4, and examine the curves for sharp peaks or sharp wiggles. (It already has been pointed out that the sharp peak at $P$ in Fig. 4(b) indicates instability.) Another way is to examine the tabulated values of $Z_a$. If any of these shows a tendency to fluctuate from interval to interval, instability is present. In Table 2 the values of $Z_a$ fluctuate in this way in intervals 12, 13.
 artisans; the illustration does not continue; therefore it is not continued unless great accuracy is desired. The occurrence of such a numerical error has been made, or that it was made, instability may occur suddenly in the middle of a wave, especially if comparatively large external forces or "shock" are inserted suddenly, or are required to change sign in the case of the third and fourth of Fig. 3.

L. and Time Interval. \( \Delta t \). In some problems the unit \( T_n \) is divided more or less automatically, as in a short railroad where each car and coupler normally would be chosen for length. If the object is continuous it may be divided into a number of units (preferably of equal length) depending on the degree of accuracy desired and the computing personnel available.

If the unit lengths have been decided, the time intervals to be used in the problem are also determined. An unnecessarily small time interval involves a great deal of extra work with little or no increase in accuracy. Too large a time interval produces instability and results in overaccuracy. The time interval produces instability and results in overaccuracy.

For a wave in a diagram such as Fig. 1 or Fig. 3, it has a "critical" interval which is the time that it would require for a sound wave to traverse this particular spring and its associated effects. Remembering that sound waves travel in both directions at a speed equal to \( \sqrt{E/\rho} \) where \( \rho \) equals the mass per unit length, the following formulas may be derived for the critical time interval for spring \( K_n \), which will be called \( T_n \):

- wave motion to right
  \[
  T_n = \frac{1}{12gK_n} \sqrt{\frac{W_{n+1}}{W_{n}}} = \frac{1}{96.48} \sqrt{\frac{W_{n+1}}{K_n}} \tag{13}
  \]

- wave motion to left
  \[
  T_n = \frac{1}{12gK_n} \sqrt{\frac{W_n}{W_{n-1}}} = \frac{1}{96.48} \sqrt{\frac{W_n}{K_n}} \tag{14}
  \]

The lower value of \( T_n \) from either formula [13] or [14] is the final value. Formulas [13] and [14] are applied to Fig. 3 the results are:

- wave motion to the right, Formula [13]
  \[ T_1 = 1.4316 \quad T_2 = 1/1030 \quad T_3 = \pi \]

- wave motion to the left, Formula [14]
  \[ T_1 = 1.589 \quad T_2 = 1/1455 \quad T_3 = 1/1120 \]

The value of \( \phi \) must never be less than unity; otherwise the results of the numerical calculation will be meaningless because the numerical calculation will not progress as fast as the actual sound wave. If the value of \( \phi \) is close to unity, instability is likely to occur. The value of \( \phi \) used for Tables 1 and 2 was \( 1.0000 = 2.00 \).

Cost of Time Interval \( \Delta t \). The time interval may be changed at any line of the numerical calculation such as Table 2. Ordinarily, this involves only the writing in of new \( \Delta \)s for the constants \( 12gK_n \) and \( \Delta g/W_{n+1} \) used in Formulas [11] and [12]. Other than this, the method must be made simultaneously in all cases \( D_n \) and \( V_n \) of the calculation, if Table 2.

Type of Impact. Practical problems may involve at least two distinct types of impact as listed and discussed in the Fig. 2. In this discussion the assumption is made that all cases are perfectly elastic. Methods for taking account of plastic and inelasticity will be discussed later on.

The spring that receives the first impact, such as \( K_i \) of Figs. 1 and 2 of Fig. 10, will be called the "impact spring." In some impact problems there is an actual impact spring or cushion on which the ram may strike. For instance, a locomotive bumping into a line of freight cars first encounters couple, spring, or "shock" gears, and must produce them in order to move the first car. Such problems will be called "cushioned impact." On the other hand, if a ram strikes a steel rod or anvil directly, there is no true impact spring present. Such problems will be called "direct impact."

A single moving weight, such as a locomotive without cars, or the moving part of a forging hammer, will be called a "ram." A group of objects, such as a railroad train in which each comparatively rigid car is separated from the next by a spring coupler, will be called "a group of separate weights and springs," and must be distinguished clearly from a "long object" such as a long steel rod or a driven pile.

An Important Distinction. In this method a long object is represented conventionally by the same type of diagram that is used for a corresponding group of separate weights and springs. If the impact is well cushioned by a comparatively "soft" impact spring, a long object and a corresponding group of separate weights and springs will act almost exactly alike. On the other hand, if the impact is direct, or if the impact spring is comparatively stiff, surges or minor oscillations may occur in a group of separate weights and springs which would not be present in a corresponding long object. This difference calls for the observance of certain rules as listed in the following section.

Types of Impact

Type I Impact. Cushioned impact between a ram and a group of separate weights and springs: This type of impact presents a diagram similar to Figs. 1 and 3. A locomotive bumping into a number of railroad cars with spring couplers is an example.

Good accuracy is obtainable with this type of problem because the diagram used is likely to be an accurate representation of the actual objects involved. The numerical calculation is similar to Table 2. The accuracy may be increased by decreasing the interval. Ordinarily, a value of \( \phi \) of 3 or 4 will give results accurate within a few per cent if the calculation is not carried through more than 100 intervals.

Type II Impact. Cushioned impact between two groups of separate weights and springs: A problem of this type presents a diagram such as Fig. 10.

![Fig. 10](image)

The numerical calculation is handled as in Table 2 except that positive velocities appear on line \( O \) for \( W_n \), \( W_i \), and \( W_f \) instead of for \( W_f \) only. Accuracy obtainable is about the same as for Type I impact.

Type III Impact. Cushioned impact between a ram and a long object: After the long object has been divided as in Figs. 5, 7, and 8, this type of impact presents a diagram similar to Fig. 1 or Fig. 3. The driving of a steel or concrete pile into the ground is an example of this type of impact, because ordinarily a cushion of wood or similar material is used between the ram and the pile. Suitable ground resistance or resistance is introduced and the problem is handled very much like the problem of Fig. 3.

The accuracy obtainable with this type of problem is ordinarily not quite as great as with problems of Type I or Type II unless the impact spring is comparatively soft. This leads to the following practical rule:

Rule. The critical time interval for the impact spring as calculated by Formulas [13] or [14] should be at least 1.5 times as large.
TRANSACTIONS OF THE ASME
AUGUST, 1953

In calculating the instantaneous velocities of \( W_1 \) and \( W_2 \) immediately after impact occurs may be calculated from Newton's law, considering these two weights only, and these resultant

\[ K_n = \frac{W_{1-n}}{2 \phi_1 \Delta t} = \frac{W_{1-n}}{360.04 \phi_1 \Delta t} \quad \text{(15)} \]

[Diagram: Fig. 11]

\[ K_n = \frac{W_{1-n}}{12 \phi_1 \Delta t} = \frac{W_{1-n}}{360.04 \phi_1 \Delta t} \quad \text{(16)} \]

so as to make \( K_n \) from either Formula (15) or (16) is the max. and a value of \( \phi_1 \) of 1.2 is recommended for the imaginary spring mentioned.

The numerical calculation the value of \( C_1 \) may become

\[ C_1 = \frac{W_{1-n}}{360.04 \phi_1 \Delta t} \]

When this happens it means that \( W_1 \) and \( W_2 \) have

borne the impact: therefore \( F_t \) should be given the value of zero until \( C_1 \) again becomes positive.

With this type of problem the accuracy may be increased by decreasing the time interval \( \Delta t \). The smaller \( \Delta t \) is made, the surer will be the imaginary impact spring as calculated from Formula (15) or (16), and the higher will be the values obtained for \( F_t \). From this it follows that the values for \( F_t \) and \( C_1 \) given by the numerical calculation are not correct or true values.

Sometime a impact spring may be obtained less arbitrarily by borrowing elasticity from the weights involved as explained in connection with Fig. 4. Even if this is done the values given by the numerical calculation for \( F_t \) and \( C_1 \) will have doubtful accuracy.

If true values must be had for the forecast or near the point of impact and immediately following the beginning of impact, they must be obtained by other means. A suitable method is explained in connection with Type VII impact. Formulas (15) and (16) are used to determine the spring constant for the imaginary impact spring as in Type VII impact.

This method has some important shortcomings as follows:

The numerical values of \( C_1, F_t, Z_t, D_t, C_t, F_t, Z_t, \) and \( F_t \) are likely to be quite incorrect. In the numerical calculation \( F_t \) serves merely as a means of transmitting energy and momentum, and its motion as given by the calculation is not likely to be correct. Numerical values of \( F_t \) and \( V_t \) starting with \( F_t \) and \( V_t \) will show peak values that may be as much as 25 per cent above the theoretically correct ones, and will tend to oscillate above and below the theoretically correct values. If halving the unit length \( l \) and making a corresponding change in the interval \( \Delta t \) and in the stiffness of the imaginary impact spring as per Formulas (15) and (16) results in more rapid oscillations, it may be concluded that these minor oscillations do not, in fact, exist, and smooth curves may be plotted eliminating them. Similar minor oscillations show up clearly in the velocity curves of Fig. 4. In the case of Fig. 4 there are true oscillations because \( W_1 \) and \( W_2 \) are separate and distinct weights. However, if \( W_1, K_1, \) and \( W_2 \) were intended to represent a single long object, these minor oscillations would have to be investigated further in the manner explained in the foregoing.

In applying this numerical method to Type VII impact it should be borne in mind that a practical standpoint its tendency to produce large minor oscillations, and thus give peak stresses and velocities higher than theoretical, is not entirely a disadvantage because of the uneven stress distribution throughout a cross section that is almost sure to occur with this type of impact. If values closer to theoretical are desired for force and velocities, they may be obtained at the cost of considerable extra computation by using Method 2 listed under Type VIII impact which follows.

Type VII impact has been discussed by Dannell (2), and a knowledge of his method is very helpful.

Type VIII impact. Direct impact between two long objects. Method 1. This type of impact may be handled like Type VI except that the two long objects first must be divided into weights and springs in accordance with Figs. 4, 5, 6, 7, and 8. The remarks as to accuracy of results made in connection with Type VII apply also except that the subscripts must be changed to
If $K_a$ is the imaginary impact spring then the values that are definitely incorrect are as follows:

\[
\begin{align*}
B_a &= C_a, & F_a &= X_a = Y_a = Z_a = V_a = W_a = 0 \\
B_a &= C_a, & F_a &= X_a = Y_a = Z_a = V_a = W_a \\
C_a &= F_a, & Z_a &= X_a = Y_a = V_a = W_a \\
C_a &= F_a, & Z_a &= X_a = Y_a = V_a = W_a
\end{align*}
\]

Other values tend to show the same minor oscillations as noted in Type VII.

Method II. If both long objects are divided into unusually small lengths, the imaginary impact spring may be chosen in accordance with the following rule without introducing any imaginary amount of elasticity into the system:

- The value chosen for $\phi_a$ for the imaginary impact spring should be 1.5 times as great as the largest value of $\phi_y$ for the object in the system. This rule is the same as the rule given under Type III impact, but is stated in different words.

- If the foregoing rule is followed, the false minor oscillations noted under Type VII impact will be almost completely eliminated, and the forces and velocities will be very close to the correct ones. The smaller the unit lengths used, the greater the accuracy.

Method II also may be used to determine accurately the forces at the point of impact in Type V and Type VI problems. Type VIII impact has been discussed by Donnell (2) and a discussion of his method is very helpful.

IRREGULARITIES

Various irregularities may be introduced into the calculation, as those illustrated in Table 2 where the resistances $R_a$ and $B_a$ are of negative sign whenever the corresponding quantity changes direction of motion, and where $F_a$ and $C_a$ become zero and remain zero. The following formulas [3] into the negative range. Sudden changes such as the foregoing are called “boundary conditions.” It is possible to devise an almost endless number of irregularities that can be handled successfully by this numerical method, but, in general, it will pay to introduce only such irregularities as will improve the accuracy of the calculated results, because the computer time involved may be considerable. Examples of various types of irregularities follow.

\textbf{Irregularities:}

1. 

- **Resistance, $R_a$.** In some problems the external resistance $R_a$ may vary from interval to interval according to some rule. For instance, $R_a$ might vary linearly with the distance $x$. In this case $R_a$ would be computed for each time interval and listed in a separate column, using the formula

\[
R_a = \psi x_a
\]

where $\psi$ is a constant.

- **Spring**. Various types of damping can be handled successfully as in Table 2. Formula [17] is also a form of damping. Hysteresis loop suitable for a spring that acts both in compression and in tension can be produced by using the following formula instead of Formula [3]:

\[
F_a = K_a \left[ C_a + \theta \left( \frac{C_a - F_a}{\Delta t} \right) \right] \quad \text{[18]}
\]

where $\theta = a$ is a suitable constant.

A hysteresis loop suitable for a spring that acts only in compression can be produced by using the following formula instead of Formula [3]:

\[
F_a = C_a K_a \left[ 1 + \theta' \left( \frac{C_a - F_a}{\Delta t} \right) \right] \quad \text{[19]}
\]

where $\theta' = a$ is a suitable constant.

\textbf{Coefficient of Restitution}. Occasionally materials, such as rubber, which have a low coefficient of restitution, are used as springs or “cushions.” Such materials when tested statically in the laboratory give force-deflection curves of the type shown in Fig. 12.

![Fig. 12](image)

The compression part of the curve may be straight enough so that the use of Formula [3] is justified up to the maximum point $F_{max}$, $C_{max}$. Fig. 12. For restitution, an equation is needed whose graph passes through the origin and the points $F_{max}$, $C_{max}$, and whose shape is similar to the restitution part of the curve shown in Fig. 12. Equation [19] satisfies these conditions reasonably well.

\[
F_{max} = \left( \frac{F_{max}}{C_{max}} \right)^u (C_a)^\eta
\]

in which $u = (2, \eta) - 1$, $F_{max} = \text{force exerted by spring m in time interval a during restitution (or recoil)}$, $F_{max} = \text{maximum value of } F_a$, $C_{max} = \text{maximum value of } C_a$, $\eta = \text{coefficient of restitution for spring m}$

During restitution only part of the energy of compression is returned. This energy return varies as the square of the coefficient of restitution $\eta$, and is represented by the shaded area in Fig. 12.

In making the numerical calculation the force $F_a$ is calculated by means of Formulas [3] until the value of $C_a$ begins to decrease. At this point in the calculation the maximum value attained by $F_a$ and $C_a$, namely, $F_{max}$ and $C_{max}$. Fig. 12, are substituted in Formula [19] and this formula is then used to compute $F_a$ until $C_a$ returns to zero.

A somewhat awkward problem arises as to what to do if $C_a$ again begins to increase before returning to zero. If only a slight hesitation is involved it is better to stick to Formula [19] until $C_a$ does return to zero, rather than to make additional changes of formula.

\textbf{Plastic Flow.} The force $F_a$ calculated by Formula [3] may become so great that plastic flow occurs, as in a forging process or
When a rule is driven into the ground. This condition corresponds to a deflection curve or the type shown in Fig. 13.

During elastic compression, formula (3) would be used. During plastic flow, formula (2) may remain practically constant. As soon as the spring begins to expand, it is indicated by a decrease in the value of \( C_m \). Formula (20) would be used:

\[
F_s = K_{m} C_m - P
\]...

[20]

where

- \( P \): amount of plastic flow, in.
- \( C_{m} \): value of \( C_{m} \) at yield point, in.

In the numerical calculation it may be convenient to list the computed values of \( C_m = C_{m} \) in a separate column.

**Graphical or Tabular Relationships.** Sometimes it is not possible in convenient to express the relationship between \( F_s \) and \( C_m \), as in formulas such as Equation (3), (8), (9), or (20), but graphs or tabular data may be available showing the relationship.

If the numerical calculation is performed by hand, the values for \( F_s \) corresponding to each value of \( C_m \) can be read directly from the graphs or obtained from the tabular data by interpolation. This process may involve the complication that the curve of restitution may have to be adjusted so that it will start at the maximum point reached by the curve of compression, as was done with a curve in Formula (19). Some electronic digital calculators can perform a similar operation based on tabular data fed into the machine ahead of time.

**Grades.** In problems involving vertical motion, the static forces due to gravity or buoyancy may be neglected. If the weights are large, as in a forging hammer, the gravity forces may be considered as negative resistances \( F_1 \), with consequent initial compressions \( C_1 \) and initial forces \( F_0 \) which must appear on line \( O \) of the numerical calculation. Alternatively, the static effects of gravity may be added algebraically after the numerical calculation of the dynamic forces has been completed. In the latter case, the following rule should be observed in making the dynamic calculation:

Rule. If any particular spring \( F_1 \) is not designed to take tension, nevertheless if \( C_m \) becomes negative \( F_t \) also must be allowed to become negative, but its negative value must not be allowed to exceed the positive static force \( F_1 \) caused by gravity alone.

When, as a final step, the static-gravity forces are added algebraically to the dynamic forces, the negative and positive values of \( F_m \) may cancel each other having a net value of \( F_m \) equal to zero.

**Branched Systems.** A simple branched system is shown in Fig. 14. Such systems may be handled readily by this numerical method. Values for \( D_m, C_m, F_m, Z_m, \) and \( V_m \) may be calculated for all weights and springs of Fig. 14 by Formulas (1) to (5), with the exception of values for \( Z_1 \). There are four springs acting on \( W_1 \); therefore \( Z_1 = F_1 - F_1 - F_4 - F_5 \). Similar adaptations of the method may be made for more complicated systems.

**Checking the Numerical Calculation.**

Plotting computed values, as in Fig. 4, is an excellent way of checking accuracy. An error will show up as a sudden irregularity in the curves. The speed with which the actual stress or sound wave progresses also will be observable, and in some problems this may be used as a check.

Repeating the numerical calculation with a smaller time interval or a smaller unit length is another method of checking.

The simplest numerical check is based on the "law of conservation of momentum," which states that the total momentum as given by the following expression always must remain constant:

\[
\text{Total momentum} = \sum W_n V_n a + \sum F_{a} \]

In using this check, \( R_{a} \) must be interpreted to mean all external forces such as \( R_{a}, R_{b}, \) and \( F_{a} \) of Fig. 3. The total momentum may be computed for any line of the numerical calculation such as Table 2.

A more complete numerical check is based on the "law of conservation of energy," which states that the total energy as given by the following expression must always remain constant:

\[
\text{Total energy} = \sum \frac{W}{2} (V_n)^2 + \sum \left( \frac{C_m}{12} \times \frac{F_2}{2} \right) + \sum \text{energy lost externally or as heat} \]

The necessary values for substitution in this formula will be found in a tabulation such as Table 2, but some of them may not be at once obvious. This energy check is not as accurate as the momentum check, especially for the first four or five intervals. Usually a check of this kind is made only at the end of the calculation.

**Acknowledgment.**

Thanks are due to Mr. William Heising, Assistant Manager, Electronic Data Processing Service, International Business Machines Corporation, New York, N. Y.; and to Dr. E. W. Milne, Head of the Department of Mathematics, Oregon State College, Corvallis, Ore., for reading the manuscript and for helpful comments; also to my son and daughter for frank criticism and material help.

**BIBLIOGRAPHY.**

5. "Longitudinal and Torsional Impact in a Uniform Bar With a
Appendix

The following notes will be of interest to those who wish to investigate fundamentals:

1. Formulas [1] to [5] are extrapolation formulas. Other formulas, as well as iterative procedures, may be devised, but formulas [1] to [5] are simple, surprisingly accurate, and have an inherent and important tendency to produce stability even when comparatively large time intervals are used.

2. A single weight and spring will obey the laws of harmonic motion. Any values may be assumed, but the simplest case for purposes of calculation is arrived at as follows:

   Emitting subscripts as unnecessary, let D, C, and K be expressed in the same units as F, Z, and V, thus eliminating the constant 12 from Formulas [1], [13], [11], [15], [16], and [22].

   Let W = w and let K = 1. Then Formulas [1] and [5] become

   \[ D = d + r \Delta \] \[ V = r - D \Delta \] \[ r = \frac{\sin \alpha}{\phi} \frac{2\pi}{2N} \]

3. From these formulas we readily may calculate values for sin \( \alpha \) and cos \( \alpha \) by letting \( V = 1,000 \) at \( n = 0 \), and considering that is measured in radians. Formula [14] then gives \( T = 1 \). If \( \phi = 5 \), then \( \Delta = 0.2 \), and it will be found that the calculated values of \( D \) approximate the tabular values of \( \sin \alpha \). However, the calculated values of \( V \) will not closely approximate the tabular values of \( \cos \alpha \) after the first interval unless the angles used in looking up the tabular values of \( \cos \alpha \) are taken as \( 1/2 \Delta \), \( 1/2 \Delta \), \( 2 \Delta \), \( 2 \Delta \), etc., which indicates that the values of \( V \) tend to be "out of phase" by about half an interval.

4. A "classical" impact problem is that of an infinite weight or ram striking directly on the end of a uniform rod of infinite length. If this problem is represented as in Fig. 1, with all springs and all weights equally spaced, exerminating the ram \( W \), then a value of \( \phi = 1 \) will give results that agree exactly with theory. Forces \( F \) and velocities \( V \) for each weight and spring will jump instantaneously to their maximum and theoretical values, and the vibration will progress down the length of the rod with exactly the speed of sound. This result will be obtained no matter what value of \( n \) is used. If, however, a larger \( \phi \) value is used for \( \phi \), the values of \( F \) and \( V \) will oscillate above and below the values obtained with \( \phi = 1 \), but, nevertheless, the peak values of \( F \) and \( V \) will travel down the rod with a speed closely approximating the speed of sound. If \( \phi \) is given a value of about 3 or more, the identical results will correspond closely to the motion of a group of oscillators, and equal, weights and springs.

Discussion

W. E. Harrigan

The equation of motion for free longitudinal vibrations in a continuous bar is

\[ m V_n = (KD) a_n = 0 \]

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where

- \( D \) = longitudinal displacement of a thin disk from its reference point
- \( m \) = mass per unit length
- \( K \) = "spring constant" or Young's modulus times cross-sectional area
- \( x, t \) are independent variables for position and time, respectively, with subscripts denoting partial differentiation

This partial differential equation is approximated by the difference equation

\[
\begin{align*}
   m(\Delta t)^2\left[D(x, t + \Delta t) - 2D(x, t) + D(x, t - \Delta t)\right] \\
   = (\Delta x)^2\left[K(x + 0.5\Delta x)D(x + \Delta x, t) - D(x, t)\right] - (K(x - 0.5\Delta x))D(x, t - \Delta t) \]
\end{align*}
\]

If the \( D \) are known for two successive time steps (equivalent to known initial positions and velocities) the foregoing difference equation can be integrated one time step at a time into the future. It is instructive to consider the simple case of a uniform bar with \( c = (K/m)^{1/2} \). The solution of the partial differential equation is

\[ D = f(x - ct) + f(x + ct) \]

In this example with fixed ends the solution of the difference equations with \( N \) space intervals yields the correct characteristic modes of sine waves traveling at a velocity \( V \). We find, however, that instead of the velocity being independent of the mode (frequency), that

\[ c = \frac{\sin \phi}{\phi} \frac{2\pi}{2N} \]

where

\[ \phi = \Delta x/(c\Delta t) \quad n = 1, 2, \ldots, N - 1 \]

We have thus the surprising result that if \( \phi = 1 \), i.e., \( c = \Delta x/\Delta t \), the solution of the difference equation is an exactly exact solution of the partial differential equation with \( c = \phi \) for all modes. If \( \Delta t > \Delta x/\phi \), \( \phi \) becomes imaginary for some large \( n \), indicating exponential behavior in time of the high-frequency modes. These cause the amplitude to increase without limit; thus the numerical method is unstable.

If \( \Delta t < \Delta x/\phi \) gives a less accurate solution to the case of the uniform bar, the highest frequencies traveling near \( 2\pi \) to \( 6\pi \) per cent of the true sonic velocity. This leads to dispersion. If \( \Delta t > 0 \), the solution approaches the behavior of \( N \) discrete identical springs. In all the stable solutions energy and momentum are conserved. In the actual physical problems, the author considers round bar, friction. The significance of \( \phi \) is qualitatively explainable in terms of the behavior found for the uniform bar. Where there is a bar striking a continuous bar, an effective spring is inserted between them. In order that the solution of the difference equations be properly, we wish \( \phi \) of the \( \phi \) approximately equal, but not less than unity. Where the bar is centered or "soft," these conditions can be made to hold readily with quite practical values for \( \Delta x \) and \( \Delta t \).

Some consideration has been given to the use of alternative numerical integration formulas. The application of the author's formulas to the case of a body in simple harmonic motion gives an apparent frequency higher than the true frequency by a factor of

\[
\left[1 + \frac{1}{24} \left(\frac{2\pi \Delta t}{\phi}\right)^2\right]
\]

Investigation shows that if \( \Delta t \) is chosen so that less than 30 per...
In his paper, the writer has selected a simpler problem which is easier to handle, and also for which the correct solution can be found so that the approximate results can be checked. The problem is this: A unit mass is drawn toward the origin by a force equal to four times the distance from the origin and subject to a resisting force of unit magnitude. The equations of motion for this problem are

\[ \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{F}(\mathbf{r}) \]

(+ if \( \mathbf{v} \) is negative; - if \( \mathbf{v} \) is positive). We take \( \Delta t = 0.1 \) and start with \( \mathbf{v} = 0, \mathbf{r} = 1 \) at \( t = 0 \).

Method 1. This is the method used by the author and is based on the equations (cf. Equations [1] and [5] of the paper)

\[ D = d + v\Delta t \]

\[ V = r + z\Delta t \]

The solution using these equations is shown under Method 1 on the accompanying computation sheet. Comparison with the true values shows fair agreement for \( D \), but poor agreement for \( V \).

Method 2. This is similar in spirit to Method 1, but uses the equations

\[ D = d + V\Delta t \]

\[ V = r + z\Delta t \]

Here the error in \( D \) is about the same as for Method 1, but \( V \) is somewhat better, mainly because the effect of the sign change in the resistance is taken into account one step earlier.

Method 3. This is not a practical computational method but is given for illustration. The values are obtained by using both Method 1 and Method 2, and then taking the average.

Method 4. This is the practical way of carrying out Method 3

### Computation Sheet

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<td>0</td>
</tr>
<tr>
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</tr>
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<tr>
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<tr>
<td>0.9</td>
<td>0</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ D = d + v\Delta t \]

\[ V = r + z\Delta t \]

\[ z = \frac{dV}{dt} = -\mathbf{v} + \mathbf{F}(\mathbf{r}) \]
far off the result may be! (b) In using Method 5 we do not need to worry about the stability of our numerical solution. For the other methods an improper choice of the interval $\Delta t$ may cause the process to diverge. In Method 5 it is easy to see whether the process converges. Finally, Method 5 gives much more accurate values of $V$, and, although we are not interested in $V$ itself, the place where $V$ changes sign is very important, as we see by comparing Methods 1 and 2. Hence we need to have reasonably accurate values of $V$.

**Author's Closure**

The discussions by Dr. Milne and Mr. Heising are important contributions to our knowledge of this subject. Mr. Heising gives mathematical background that is not included at all in my paper, and which should prove valuable to anyone who wishes to investigate the problem thoroughly. Dr. Milne proposes certain alternative methods for the purpose of increasing accuracy.

Dr. Milne's Methods 2, 3, and 4 are basically identical with my method (which he calls Method 1) except in the handling of the frictional force. The formula he uses for $D$ in Method 4 is a combination of the two formulas he uses for my method. This combination can easily be made if it is remembered that in Dr. Milne's notation $V = n^* + \gamma \Delta t$, and $V^* \Delta t = d^* - \gamma^*$, where $d, n, \gamma$, and $\gamma$ indicate values in time interval $n - 1$, and $d^*$ and $\gamma^*$ indicate values in time interval $n - 2$. Dr. Milne's problem involves rapid frictional damping applied directly to the ram. Accurate results on this problem can be obtained by my method by using a smaller time interval.

Dr. Milne's Method 5 uses a correction formula to check or correct each step in the calculation, and is designed to give accuracy specifically on problems involving the idealized condition of springs entirely without weight and weights entirely without elasticity. My method is suitable for such problems if comparatively high values of $\phi$ are used, and for problems involving partially or completely distributed weight and elasticity if lower values of $\phi$ are used.

In closing it may be well to point out that torsional problems can be handled in a very similar manner.

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1 Author's reference (7), articles 8, 9, 10, and 11, pp. 24-30.
What Happens When Hammer Hits Pile
Now it can be told:

Up to now no one had been known to solve manually the complicated wave equation as it applies to action of a pile under the blow of a hammer. Today it takes but seconds to solve "mechanically" for driving resistance information such as plotted here.

Electronic digital computers are solving the wave equation to answer the question...

What Happens When Hammer Hits Pile

Edward A. Smith
Chief Mechanical Engineer
Raymond Concrete Pile Company

It is now possible, thanks to electronic digital computers, to calculate in a practical and economical manner what happens at the instant a hammer hits a pile. As a result, it is possible to do with far more accuracy than ever before that a certain type pile, driven into a particular soil to a determined length of blows per inch, is being driven against a definite number of load units.

Electronic computers are used to determine the safe bearing capacity of a pile. Exceptions occur with piles that either "set up" or "rattle" when driving, so a knowledge of 2 and 4 characteristics is essential, but in the main, the ability to compute the driving resistance accurately will greatly improve the engineer's position in determining the safe bearing value of a pile.

All this is now possible because electronic digital computers are available to solve in numerical methods the difficult mathematical involved in applying the "wave equation" to pile driving problems.

- The Wave Equation—The wave equation itself is not new—it has been known to mathematicians for many years. In general, it describes how waves progress from one point to another; specifically, it may be used to illustrate the wave action produced in a long object by a force suddenly applied at one end. In his book "Theory of Elasticity" (McGraw-Hill, 1954), Timoshenko showed how the wave equation might be used to calculate longitudinal wave action in a uniform beam.

Furthermore, the idea of applying the wave equation to pile driving is not new. In 1940 and in 1941, A. E. Cummings referred to the subject, and credited D. V. Isaacs of Australia with being, in 1951, the first to suggest its application to pile driving. Cummings also stated that the British Building Research Board in 1958 demonstrated that the behavior of full sized piles under actual field conditions can be predicted by means of the wave theory.

Yet, the wave theory has not, until very recently, been applied to pile driving. The reason for this is simple, and as given by Cummings—the calculations involved were too difficult. As given by Timoshenko, the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$$

is a second order partial differential equation. For very simple cases, as
when a known force is suddenly applied at one end of a uniform steel rod, the equation can be solved by ordinary calculus. But when the equation is complicated by considerations of the actions of the ram, the cap block, the pile, and the ground, the problem becomes so difficult that no one has been known to solve it.

- Computers to rescue—However, the picture has been changed in the last few years by the development of electronic digital computers such as the Sperry Rand “Univac” and the IBM “704.” Not only can these machines solve the wave equation as applied to pile driving—but they can do it in a matter of seconds.

The Raymond Concrete Pile Co., working in conjunction with IBM, has made over 250 such computations involving various combinations of piles and hammer types. The mathematical methods used have been checked in the works and proved reliable. From the results of these calculations, the writer will draw some conclusions, subject to the following qualifications:

1. The importance of variation in ground characteristics as explained in a previous paragraph must be considered as a limiting factor in arriving at definitive conclusions.

2. In applying the wave theory to pile driving, it is possible, with the computers, to include side friction as part of the calculation and in the immediate future this will be done, but thus far the calculations have been made on the basis of an end bearing pile with no side friction.

3. It is not the intent of the writer to offer a new and different pile formula. The problem is too complicated to be expressed accurately by a single reasonably simple formula. Possibly a formula could be devised with certain constants that would be changed to suit different types of driving conditions, but, for now at least, the results of the calculations are presented only in graph form.

- Conclusions—Graph No. 1 shows ultimate driving resistances for various lengths and weights of straight-sided steel piles driven with a No. 1 Vulcan hammer. For comparative purposes the ENGINEERING News formula is also plotted both in its ordinary safe load form and also in its ultimate resistance form based on the fact that this formula includes a factor of safety of 2.5. It will be noted that the wave equation curves show considerable variation from one another and indicate that use of the ENGINEERING News formula results in a factor of safety that varies from a little more than one to as much as four.

Graph No. 2 shows a comparison of the ENGINEERING News formula, the Hiley formula and the wave equation for step-taper piles. It will be noted that in general the wave equation indicates that step-taper piles can take quite high loads, much more than allowed by the Hiley formula. It must be remembered that the Hiley formula can be made to give widely varying results depending upon what constants are used with it. In preparation of the graph, therefore, the constants used corresponded to those used with the Wave Equation.

Graph No. 3 shows a comparison between the ENGINEERING News formula and the Hiley formula.
A Hiley formula and the wave equation, for No. 1 and No. 2, agree in general the wave equation and Hiley formula for a light pile, a 4.5.

For No. 3 and No. 4, however, the wave equation shows the formula to agree with the Hiley formula and to give the correct assumptions for the pile is entirely incorrect. This is the condition of the pile is entirely incorrect. This is the condition of the pile is entirely incorrect. This is the condition of the pile is entirely incorrect. This is the condition of the pile is entirely incorrect.

No. 5 shows a comparison on a site load basis between the wave equation with a factor of safety of 3 and certain formulas employing the recommended or usual factors of safety.

From all these data it must be concluded that the Hiley formula tends to penalize the heavy or long pile and to favor the light and short pile. This occurs because the Hiley formula is shown by the Wave Equation calculations to contain incorrect assumptions as follows:

1. It assumes that the compression of the cap block, the compression of the full length of the pile, and the elastic compression of the ground all occur at one and the same instant of time. This is reasonably accurate for a light pile, but inaccurate for a very heavy pile.

2. It assumes that the percentage of energy transmitted from the hammer to the pile is dependent on the relative weights of the pile and the entire pile. This assumption is reasonably accurate for a short pile, but inaccurate for a long pile.

Chelli's in his book "Pile Foundations" (McGraw Hill, 1951) lists 30 pile formulas that have been proposed and used. No one of them can be accurate for all types and lengths of piles because they all fail to consider the effects of wave action. The newly acquired ability to solve the wave equation as applied to pile driving offers a means of obtaining a truly mathematical solution.

In due course it is expected that additional calculations will be made covering a variety of hammer and pile types, and that the results will be published in the form of tables, resembling logarithmic and trigonometric tables. Meanwhile, it is hoped that the graphs here presented will prove both informative and provocative.
THIS CALCULATION, performed in the IBM Technical Computing Bureau, provided a complete numerical analysis of the behavior of a pile when struck by a pile-driving hammer. The results indicated what stresses occurred within the pile and capblock, as well as the penetration of the pile into the earth. This is an example of the replacement of costly experimentation by economical calculation.

Previously, using only key-driven calculators, it has been possible to study only partially a few isolated cases of the behavior of a pile under an impact. This was due to the high cost in time and money associated with each case studied. By employing the high speed and accuracy of IBM electronic calculating machines to perform this repeated formula evaluation, it is possible to study the behavior of numerous pile types. The cost of each case studied thereby will be substantially reduced.

When piles are driven as a foundation for a building or other structure, the load that they will carry safely usually is determined by measuring the penetration under the last blow of the hammer and substituting this figure in a formula. Many different formulas are in use, and they vary widely in the answers they give. This calculation provides a means of mathematically testing the accuracy of these formulas when applied to various types of piles and various ground conditions.

Furthermore, when the engineer has been asked what stresses the pile or the pile-driving equipment should be designed to withstand, he has often been at a loss for an answer. He has known only that certain practices and equipment have proved successful in the past. To do something new has meant proceeding by the often costly process of trial and error.

Because a pile is an object of considerable length, pile driving is a problem in longitudinal wave transmission and impact, the basic principles of which were first investigated 100 years ago by the French mathematicians Saint Venant and Boussinesq. These principles have been ably set forth by L. H. Donnell in ASME Transactions APM-52-14. However, the problem is a very complicated one. The method of solution offered herein is based on approximate integration using a step-by-step calculation, and is of general interest because it can be applied to other impact problems. The calculation can be done by hand with a slide rule or desk calculating machine; however, modern electronic digital calculating machines are especially well adapted to make the calculation quickly and easily.

**General Case—Basic Theory**

For purposes of analysis, the hammer, pile, etc., will be represented by a series of concentrated weights separated by weightless springs. Subscripts \( n-1 \), \( n \), \( n+1 \), etc., will be used to denote order of position, and superscripts \( n-1 \), \( n \), \( n+1 \), etc., will be used to denote order of time.

Referring to Figure 1, let \( W_{n-1} \), \( W_n \) and \( W_{n+1} \) represent successive weights, and \( S_{n-1} \), \( S_n \) and \( S_{n+1} \) the springs underneath the respective weights. Let the initial positions

**Figure 1**
of weights $W_m$, etc., at the beginning of impact be indicated by $O_m$, etc., and the initial lengths of the springs, by $L_m$, etc. Also, let

- $L_m$, etc. = instantaneous displacements (inches).
- $L_m$, etc. = instantaneous lengths (feet).
- $C_m$, etc. = $12[L_m^2 - L_m] = D_m - D_m + 1 = instantaneous amount of spring compression (inches).
- $K_m$, etc. = elastic constants for springs $S_m$, etc. = force required to produce $1"$ of compression $C_m$, etc. (pounds per inch).
- $F_m$, etc. = instantaneous forces (pounds) resulting from $C_m$, etc.
- $V_m$, etc. = instantaneous velocities of $W_m$, etc. (fps).
- $R_m$, etc. = external forces, such as ground resistance, affecting the motion of $W_m$, etc. (pounds).
- $Z_m$, etc. = instantaneous net force acting on $W_m$, etc. = $F_m - F_m - R_m$, etc. (pounds).
- $t$ = time (seconds).

Also, let superscripts $n-1$, $n$, $n+1$, etc., denote successive time intervals $\Delta t$, so small that with negligible error it may be assumed that all forces and velocities remain constant during each time interval. Then, if by previous calculation or otherwise, the values of $V$ and $D$ are known for some particular time interval $n-1$, all values for $C$, $F$, $Z$, $V$ and $D$ for the next time interval can be calculated as follows:

Let $V_{n-1}^n$ and $D_{n-1}^n$, etc., represent definite known values at time interval $n-1$, and let $C_n^a$, $F_n^a$, etc., represent definite values to be calculated for time interval $n$. Then the following general formulas apply:

$$C_n^a = D_{n-1}^a - D_{n-1}^a$$

$$F_n^a = K_m C_n^a$$

$$Z_n^a = F_{n-1}^a - F_n^a - R_n^a$$

$$V_n^a = V_{n-1}^a + \Delta V_{n-1}^a = V_{n-1}^a + (32.17 \Delta t) Z_n^a$$

$$D_n^a = D_{n-1}^a + \Delta D_{n-1}^a = D_{n-1}^a + (12 \Delta t) V_n^a$$

$\Delta V$ above is evaluated by using the standard formula for change of velocity

$$v_1 - v_2 = \frac{V}{M} \times \Delta t$$

where $W$ is the weight, and $g$ is the acceleration of gravity. $\Delta D$ above is evaluated by the formula for distance traveled $s = vt$ with the coefficient 12 introduced to convert to inches.

Careful consideration of the above formulas will disclose that the force at the beginning of an interval is used to calculate the velocity at the end of the interval, and then this end velocity is used to calculate the distance traveled during the same interval; therefore, the forces, velocities, and displacements used are slightly out of step with one another, depending on the size of the time interval.

These five formulas are used repeatedly until the calculation has been carried as far as necessary. They apply whether or not the successive weights, elastic constants, and external forces are equal or unequal. Furthermore, the values of $K$, $R$, and $\Delta t$ may be changed from interval to interval according to any definite formula, or suddenly as required. Sudden changes are required, for instance, when the stress in a material reaches the yield point, when a weight loses contact with a spring designed only for compression, when a coefficient of restitution is introduced, or when a ground resistance force must be made negative so as to resist temporary upward movements. Such conditions are called boundary conditions. Some of the more complicated digital calculators can take care of these boundary conditions automatically.

**Choice of Lengths L and Time Intervals $\Delta t$**

It should be borne in mind that formulas (1) to (5) involve the use of small but finite increments, not infinitesimals. The lengths $L$ and the time intervals $\Delta t$ must, therefore, be chosen small enough to suit each particular type of problem. For each new type of problem halving or quartering the size of the units and recalculating part of the problem must be tried until it is found that the use of smaller units makes a negligible difference in the peak stresses that occur soon after impact. The time interval can be changed while a calculation is in progress by inserting a new value for $\Delta t$ in formulas (4) and (5), although this change is subject to the limitation pointed out in Step 2 below.

**Illustrative Problem**

A pile of non-uniform section as shown in Figure 2 is to be driven through water or very soft mud to a hard layer of ground which is capable of resisting a maximum force of 600,000 pounds under the pile point. If the point of the pile starts to move upward momentarily, a negative frictional force of 100,000 pounds must be assumed, acting to hold the point down. No other side frictional forces are to be considered. The calculation is made to determine the final penetration per blow at which the assumed point resistance of 600,000 pounds will be developed.

Side friction along the pile has been omitted from this problem so as to allow the stress wave to travel with the known speed of stress (or sound) in the pile material and thus provide one way of checking the calculation. The method would apply equally well no matter what values were assigned to side friction.
Step 1. Decide on the time interval \( \Delta t \). From previous experience a time interval of 1/4000 second has been chosen as being small enough to give accuracy within about 5%.

Step 2. Decide on lengths \( L \). These must be at least as great as the distance stress will travel in the chosen time interval \( \Delta t \); otherwise the stress wave will run ahead of the calculation, and the results will be meaningless. It is recommended that \( L \) be made equal to twice the distance that stress would travel in the chosen time interval. The upper part of this pile is entirely of steel, and the known speed of stress in steel is 16,800 fps; therefore, the recommended length for \( L \) is \((16,800 \times 2) / 4000 = 8.4 \) feet. The pile length, plus a little added for the follower, happens to be a multiple of this figure; therefore, 8.4 feet can be used throughout. If an odd length were required, it would be inserted at the point of the pile.

Step 3. Prepare a diagram as per Figure 3 showing how the ram, capblock, follower, and pile are to be represented for purposes of calculation. The individual weights \( W_i \), etc., are calculated so as to give a weight distribution closely equivalent to that of Figure 2.

Step 4. Prepare a tabulation of all constants required for formulas (1) to (5) as per Figure 4. This is readily done by considering the weight, cross-section, and modulus of elasticity of each portion of Figure 2 equivalent to a single spring or weight in Figure 3. The elastic constant \( K \) for the wooden capblock must be determined by experiment or must be assumed. For this problem, a value of 6,400,000 pounds per inch has been assumed, which represents a rather stiff capblock, perfectly elastic.

Step 5. If the work is to be done by hand, it is conveniently tabulated as shown in Figure 5, which covers only the first three time intervals. In order to start the calculation, it is necessary to have a value for \( V \) and a value for \( D \) in the first time interval. The value of \( V \), representing the velocity of the ram at the beginning of
(Δt = 1/4000 Second)

<table>
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<tr>
<th>Subscript &quot;m&quot;</th>
<th>W</th>
<th>K</th>
<th>R</th>
<th>(32.17 \Delta t)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(\Delta t)</td>
<td></td>
<td></td>
</tr>
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<td>7500</td>
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<td>1,432,500</td>
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<td>0</td>
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</tr>
<tr>
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<td>7,206,000;</td>
<td>0</td>
<td>1,086,500</td>
</tr>
</tbody>
</table>

Note: \(R_{1} \) assumed = \(F_{1} \) until \(F_{1} \) reaches 600,000#. Thereafter \(R_{1} = 600,000\) if \(W_{1} \) is moving down and \(-100,000\) if \(W_{1} \) is moving up.

**Figure 4**

<table>
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<tr>
<th>(n)</th>
<th>(C_{1})</th>
<th>(J_{1})</th>
<th>(Z_{1})</th>
<th>(V_{1})</th>
<th>(D_{1})</th>
</tr>
</thead>
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<td></td>
<td>(D_{1}^{n-1} - D_{2}^{n-1})</td>
<td>6,400,000C_{1}</td>
<td>(-F_{1}^{n})</td>
<td>(V_{1}^{n-1} + \frac{Z_{1}}{932,500})</td>
<td>(D_{1}^{n-1} + .003V_{1}^{n})</td>
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<td>Pounds</td>
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</table>

**Figure 5**

<table>
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<tr>
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<th>(F_{2})</th>
<th>(Z_{2})</th>
<th>(V_{2})</th>
<th>(D_{2})</th>
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<td>7,206,000C_{2}</td>
<td>(F_{1}^{2} - F_{2}^{2})</td>
<td>(V_{2}^{n-1} + \frac{Z_{2}}{182,500})</td>
<td>(D_{2}^{n-1} + .003V_{2}^{n})</td>
</tr>
<tr>
<td>1</td>
<td>Inches</td>
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<td>Pounds</td>
<td>Ft. per Sec.</td>
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</tr>
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</table>

Note: Line 3 is not complete because the Force \(F_{2}^{2} = 32,409\) can be used to calculate values for \(Z_{2}^{2}, V_{2}^{2}\) and \(D_{2}^{2}\) which are not shown. Velocity of Ram at Impact = 14.250 f/ps.
the impact, must be calculated by considering the distance it falls and allowing for hammer efficiency. For this problem, efficiency was assumed to be 90%, which gave a velocity of 14.25 fps to be used in starting the step-by-step calculation. The numerical value of the displacement \(D\) in the first time interval is obtained from the assumption that the ram continues to move with undiminished velocity through the first 1/4000 second after the impact. For an impact velocity of 14.25 fps this gives a displacement in the first time interval of 0.04275 inches. This displacement then is used to calculate force \(F_1\) for the second time interval, and so on. As the stress wave travels down the pile, additional columns are needed in groups of five, all headed by the basic formulas (1) to (5) using constants taken from Figure 4.

If the work is done by an electronic digital calculator, the results will be tabulated by the machine as shown in Figure 6. This is a condensed tabulation which shows time intervals 1 to 5 and some of the later time intervals. For the later time intervals, only the data for the top

<table>
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<th>Subscript (m)</th>
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<th>Forces (R) (\text{pounds})</th>
<th>Velocities (V) (\text{fps})</th>
<th>Displacements (D) (\text{inches})</th>
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**Figure 6**
and bottom of the pile are included, because these are the controlling factors in the calculation.

The calculation begins in the first time interval with a velocity of 14.25 fps and a displacement of 0.04275 feet. The stress wave travels down the pile, bringing each successive section of the pile into action. For example, in the fourth time interval, the velocity $V_3$ amounts to only 1.5946 fps with a displacement of $D_3$ of only 0.00391 inches, and a total force $F_3$ of only 8100 pounds.

It will be observed that, to represent the specified ground resistance, $R_{14}$ is given a value equal to $F_{13}$ until $F_{13}$ exceeds 600,000 pounds, as it does first at time interval 30. From this point on, $R_{14}$ remains at 600,000 pounds, except that when $V_{14}$ becomes negative, as it does in time interval 56, $R_{14}$ is given a value of $-100,000$ pounds in the next time interval, which, in this case, is interval 57. In interval 57, the velocity $V_{14}$ is again positive; therefore, $R_{14}$ is again given a value of 600,000 pounds in interval 58, and so on.

In the meantime, the values of the displacements $D_{14}$ have increased gradually to a maximum of 0.28940 in interval 58, after which they remain practically unchanged. It will also be observed that the capblock force $F_3$ becomes negative in time interval 59. This means that the hammer ram has separated from the capblock; therefore, at this point, the hammer ram is dropped from the calculation.

Figure 7 is a graphic representation of the results of the calculation. A separate curve is plotted for each weight $W$.

![Figure 7: Displacements](image-url)
and the curves represent the relationship between displacement and time. It can be seen from the curve for \( N_e \) that the hammer ram curve crosses the pile head curve in the 58th time interval. On the curve for \( N_{14} \), it can be seen that the penetration reaches a maximum in the 53rd time interval and then fluctuates slightly in the succeeding time intervals, reaching practically the same maximum again in interval 58. The calculation should always be carried beyond the first maximum of penetration in order to make sure that only slight fluctuations in penetration will occur thereafter.

**Checking the Calculations**

The total energy of the system for any particular time interval can be obtained by adding the kinetic energies of the individual weights, the potential energies of the individual springs, and the total work performed in overcoming the various external forces. The total should equal the energy of the ram just before impact.

The total momentum of the system for any particular time interval can be obtained by adding the products of each mass multiplied by its instantaneous velocity and the products of each external force multiplied by the total time it has acted. The total should equal the momentum of the ram just before impact.

Neither check is exact because of minor inaccuracies in this method, but a sudden variation between one time interval and the next indicates a numerical error. If the total varies by more than about 5%, consideration should be given to reducing the time interval and possibly the lengths \( L \). If the work is done by hand, it is recommended that checks be made at every 10th interval. If the work is done by an automatic calculator, it may be possible to include a running check as part of the setup. The energy check is to be preferred to the momentum check as it is more complete. Plotting the calculated results for displacements \( D \) as per Figure 7 is also an excellent check on the reasonableness of the results. Curves for separate calculations may be readily compared.

**Recalculation for Change in Ground Resistance**

A change in the resistance near the point of the pile will change only a half or a third of the total calculation. Piles may, therefore, be recalculated for various point resistances with a considerable saving of effort as compared with an entirely new calculation.

**DISCUSSION**

*Mr. Sheldon:* I would like to make a few comments. I have been much impressed with Mr. Smith's handling and understanding of the phenomena which go into these piles. He has not employed calculus, but he has really derived for you the partial differential equation for the motion of the pile with boundary conditions.

The equation for the displacement \( D(x,t) \) in a one-dimensional, inhomogeneous elastic medium is:

\[
\rho(x) \frac{\partial^2 D}{\partial t^2} = \frac{\partial}{\partial x} \left[ Y(x) \frac{\partial D}{\partial x} \right] + f(x,t)
\]

where \( \rho(x) \) is the mass density, \( Y(x) \) is Young's modulus, and \( f(x,t) \) is the applied stress. The simplest difference system by which we can replace the above differential equation is, in the notation of Mr. Smith,

\[
\frac{D_{n+1}^m - 2D_n^m + D_{n-1}^m}{(\Delta t)^2} = \frac{Y_{n+1}[D_{n+1}^m - D_n^m] - Y_n[D_n^m - D_{n-1}^m]}{(\Delta x)^2} + f_n.
\]

The solution of this difference system is exactly equivalent to the solution of the difference system derived by Mr. Smith from first principals. To see this, note that

\[
K_m = \frac{Y_m}{\Delta x}, \quad F_n^m = K_m[D_n^m - D_{n-1}^m], \quad R_n = f_n^m \Delta x,
\]

\[
W = \frac{3\Delta x}{22.17} = \rho m \Delta x \quad \text{and} \quad V_n^m = \frac{D_n^m - D_{n-1}^m}{12\Delta t}.
\]

Mr. Smith chose an interval \( \Delta t = \Delta x/2c \) (\( c \) = sound speed), so that he had a safety factor of 2 in the Courant condition for the stability of numerical integration of hyperbolic type equations.

We solved this problem on the card-programmed calculator at the technical computing bureau. We chose the CPC for solution because Mr. Smith had quite complicated boundary conditions imposed. For one thing, the resistance of the ground is a non-linear function. It is 600,000 pounds upwards when the last weight is moving down and 100,000 pounds downward when the last weight is moving up. Another condition that has to be provided for is that the capblock and follower are not attached to the pile itself, so that after a certain period of time the capblock and follower fly off the top of the pile. Using the card-programmed calculator with its facility to list answers as we go along, we were able to observe the sign of the velocity at the bottom of the pile and also whether there was a tension or compression in the capblock. As soon as the condition which bounded this motion changed, we were able to insert a new instruction card which would take care of the new condition. This is a much simpler procedure than attempting to put these conditions into the control panel of a machine. We have solved, totally, eight cases of this pile-driving work, and for the last six cases there was an additional complication in the auxiliary conditions. Mr. Smith decided to take account of the fact that the capblock was made of wood and, therefore, was not perfectly elastic. We changed the elastic constant of the capblock according to whether the capblock was being compressed or was expanding.
The problem runs at about one step in the time every three minutes. This is an average figure, taking account of changing the program cards to take care of auxiliary con-

Dr. Aronofsky: I do not understand the boundary conditions at the top. The weight of the ram is 7,500 pounds. Is there any condition imposed?

Mr. Sheldon: The weight at the top is a freely falling weight; so the boundary condition at the upper end of the pile is that at \( t = 0 \) the ram has a certain definite velocity, and all the other weights are not moving.

Dr. Aronofsky: Is there any assumption about resistance along the lateral side of the pile all the way down?

Mr. Sheldon: In one case there was just the resistance at the bottom. In another case the resistance was applied in the middle.

Mr. Smith: In Figure 4 the only resistance that is inserted is the last one. All the other resistances are zero.

However, if desired, you can put in as many resistances as there are weights.

Dr. Buchholz: I think such studies have been made on analog equipment. You replace this type of system by a network of little capacities and provide certain nonlinear elements to take care of boundary conditions and special conditions. You run into a bit of a problem in the case of the capblock leaving the rest of the system. I don't know whether this problem has been done, but I imagine it might be possible to do so.

Mr. Moncreiff: Was special wiring used for this problem?

Mr. Sheldon: No effort was made to change the standard setup at all. We made it very simple so that it took about one day to plan for the machine.

Mr. Moncreiff: The calculation is simple enough so that you could save time by wiring a special control panel.

Mr. Sheldon: That is true.
SPECIAL OFFSHORE PILEDRIVING TEST

by Wolter R. de Sitter,
Hollandsche Beton Groep N.V.

ABSTRACT

Predictions of the bearing capacity of foundation piles generally are based upon soil investigations on site and in the laboratory. In addition, load tests on completed piles have been performed. These provide the checks on prediction methods and experimental validation of established design methods. Actual load tests on large offshore piles however, are very few and tests on piles with bearing capacities of 20 to 40 MN (4,500 to 9,000 kips) would lead to prohibitive costs. Hence an increasing importance is attached to the significance of the Soil Resistance during Driving (SRD) as a yardstick for - or at least a check on - the ultimate static bearing capacity.

The SRD may be determined by dynamic measurement of strains and accelerations in the pile and also by post-analysis of blow-counts. On a test site in the Netherlands a 68" O.D. batter pile (6:1) was driven by a Hydroblok type HBM 4000 hammer to 70 m penetration. The pile was instrumented to determine SRD. Accelerations were determined by a new strain difference technique as an alternative to conventional accelerometers. An extensive soil investigation on site and in the laboratory was included in the test program.

Post-analysis of blow-count lead to values close to the static bearing capacity at successive levels predicted according to the Cone Penetration Test (CPT) method. The SRD obtained by measurements proved to be a lower bound for the predicted static bearing capacity (80%).

Offshore field evidence from the driving of anchor piles in the Eastern Scheldt show the same correlation: between SRD obtained by post-analysis of blow counts and predicted static bearing capacity on the basis of Cone Penetration Tests, except in cases where the CPT's indicated extremely high values for cone resistance and local skin friction.

INTRODUCTION.

The delivery procedure for two Hydroblok HBM 4000 hammers included an endurance driving test. The opportunity to gather information of a more general nature on the behaviour of the interaction between soil, pile and hammer generated sufficient international support of the industry, certifying authorities and scientific organizations to include soils investigations and instrumentation of the pile in the test program. In this paper some of the results and additional field evidence are presented. The methods used to obtain SRD values and static bearing capacity are summarized and discussed. Data on the test programme and additional field evidence are introduced, followed by a summary of the results and conclusions from comparisons of SRD values and predicted bearing capacities. The objective of the paper is to contribute to the development of methods to determine axial bearing capacity of piles during the actual driving process. Controlled pile-driving presents the opportunity to collect fairly accurate information on the soils during driving. The problem needing the most attention in the future is the interpretation of this information with respect to the inservice behaviour of the piles during the lifetime of the construction.

SOIL RESISTANCE DURING DRIVING (SRD)

The theory of the measurement of the Soil Resistance during driving has been derived by several authors (references 1, 2 and 3). In its most simple form the SRD can be expressed as the sum of the downward travelling force wave \( F_1(t) \) in the pile at the time \( t \) and the upward travelling force wave \( F_2(t + 2H/c) \) at a moment \( 2H \) later. Herein \( H \) is the distance of the measurement level near the top of the pile to the toe and \( c \) is the propagation velocity of the force waves.

\[
\text{SRD} = F_1(t) + F_2(t + 2H/c)
\]

In practice we must derive the force waves from the measurement of strains and accelerations must be integrated to obtain the local velocity in the pile during a blow of the hammer.

For the measurement of strains proven strain-gauge techniques are used. The measurement of accelerations in the range of 500 g with sufficient accuracy
is still a problem, because subsequent integration in order to obtain velocities puts a high demand on accelerometers and measurement techniques. During the first part of the stress wave in the pile, when \( F(t) = 0 \), the following relationship exists between the velocity and the normal strains.

\[
v(t) = c(t) - \frac{c(t)}{c(t)} = -2(2)
\]

This relationship can be used to obtain a check on the velocities which have been calculated during the early part of the blow as long as no upward travelling stress waves have reached the measurement level. Some standard procedures use the ratio between the expected value of \( v(t) \) and the measured value as a scaling factor to correct subsequent results of the integration procedure.

A second method to obtain the SRD is post-analysis of driving records. In its most archaic (and unrealiable) form pile-driving formulas are used. Modern computer techniques however, enable us to simulate the pile driving process in a much more exact manner (ref. 1 and 4). The theory is straightforward and based upon accepted mathematical and mechanical concepts. However, the soil investigation techniques such as Cone Penetration Tests, Standard Penetration Tests and Borehole, do not lead to direct input data for the soil-pile-hammer computation model. Experience of a semi-empirical nature is needed to translate soil data from field- and laboratory-tests into calculation parameters, such as dynamic skin friction, damping, quake, and toe resistance. With conventional steam hammers the input data about hammer performance often also are questionable. The impact velocity just before the ram strikes the anvil and the loss of energy in the cushion block are often known only within wide limits. To account for these unknowns often so-called "efficiency" factors are introduced. With modern Hydroblock hydraulic hammers, the impact velocity can be controlled and a read-out of its exact value is given on the operators desk. Moreover, the built-in buffer obliterates the need for a cushion block (ref. 1 and 10). In such cases the input data for the hammer are known within narrow margins and the introduction of efficiency factors then is irrelevant (ref. 5).

A combination of post-analysis and direct measurements can be used in order to increase the accuracy of the results. It leads to iteration procedure between measured strains and velocities and post-analysis of these measured parameters until sufficient convergence is obtained (ref. 3 and 6).

What are the main considerations in the use of the described techniques to determine the SRD?

- The direct measurement depends heavily on the accuracy of the measurement of accelerations. Especially with steel (as opposed to concrete-) piles the accelerometers must be sufficiently rugged to withstand high frequency accelerations in the order of 500 g and still be sufficiently accurate to permit the calculation of the velocity by integration. In practical applications these demands pose a serious problem.

- The direct measurement provides a sum of the friction and the toe resistance which act on the pile at different times at different levels. This poses a problem as to the interpretation of the result. A value for the resistance of the soil is found, but it not totally clear what it does represent.

- The post-analysis depends on the interpretation of soil investigations. Immeasurable parameters such as damping and "quake" are involved. Given a sufficient number of such parameters answers within wide ranges can be obtained in driveability studies. However, the post-analysis treats the mechanical interactive behaviour of three components, the hammer, the pile and the soil. The presentation of the physical properties of the pile generally poses no problems. Provided that the properties and the behaviour of the hammer are well defined and controlled during driving these also can be correctly introduced. Then it is in principle possible to determine the actions of the third component, the soil, on the pile during driving (i.e. the SRD) on the basis of a series of observations of the behaviour of the total hammer-pile-soil-system. A blow-count graph is such a series of observations of system behaviour. The result of the calculation, the SRD then includes the effects of all soil parameters. The values of single soil parameters cannot be obtained by post-analysis, (nor can they be measured in the field).

In the cases under consideration in this report the impact force has been measured and the measurements agree very well with the computed impact force/time diagram at the same setting of the pretensioned buffer and the observed impact speed before the ram strikes the anvil (fig. 3). Thus the computation accurately reflects the hammer pile interaction. Therefore the SRD's obtained using the same computation process and system model in the post-analysis may be viewed with confidence.

**STATIC BEARING CAPACITY.**

The forecast of the axial Static Bearing Capacity in general is based upon data from soil investigations both in the field and the laboratory. The necessary input parameters for bearing capacity calculations are determined from tests but they cannot be measured directly. Recourse to experience of a semi-empirical nature provides the link between the results of soil tests and bearing capacity calculation procedures. Full scale load tests on completed piles form the justification and calibration of calculation techniques.

In this paper the results of Cone Penetration Tests have been used to calculate the static bearing capacity \( Q \). Methods are treated in references (11), (12), (13), (14) and (15). The point-bearing \( Q_p \) was determined using the method outlined in references (13) and (14). The skin friction \( Q_s \) was calculated from the measured local sleeve friction \( p_f(x) \) as follows:

\[
Q_f = \frac{H}{F} \int p_f(x)dx \quad \text{(Closed Toe)} \quad (3)
\]

\[
Q_f = 1.5 \cdot \frac{H}{F} \int p_f(x)dx \quad \text{(Non-plugging open-ended pile)} \quad (4)
\]

(The factor 1.5 accounts for the friction on the inside of the pile).
Then: \[ Q = \frac{Q_p}{p} = Q_f \]  

The relation between the static bearing capacity and the SRD is affected by various factors which will be discussed briefly.

- **Pore pressure.** During driving the pore pressure may increase or decrease. Thereby the effective stress may decrease or increase proportionally. Hence an increase of pore pressure may decrease the SRD.

- **Friction fatigue.** This term is used to indicate the decrease of the skin friction during driving. In reference (9) an application of the theory is described, which refers to clays only. Both remoulding of the soil and temporary reduction of effective lateral soil stresses, in particular in overconsolidated conditions, are quoted as sources of this phenomenon.

- **Damping.** Damping indicates the velocity dependence of the skin friction during driving. It leads to an increase of the apparent skin friction during driving.

  With respect to the SRD it has an effect opposite to the influence of friction fatigue. In reference (8) it is indicated that for sands with a low silt content there is no velocity dependence of the skin friction.

- **Added mass.** Acceleration of soil particles will lead to inertia forces acting on the pile. These are included in the measured value of the SRD. This effect is more important with piles with a closed toe than with open ended piles.

- **Plugging.** In most cases no plugging will occur in open ended piles during driving. (ref.9).

  Therefore analysis for the open-ended piles in the Eastern Scheldt have been performed on the basis of non-plugged piles.

- **Time effects.** With respect to the phenomena which occur during driving the service loads on piles are of a static or semi-static nature. With conventional steam hammers the duration of the effective part of a blow is in the order of 2 ms in which time the stress wave in the soil travels 3 meters. This means that in cases of large diameter piles the slip surfaces which govern semi-static failure mechanisms cannot develop during driving due to the short duration of the blow. In cases of hydraulic hammers with a pretensioned gas buffer the duration of the blow is in the order of 9 to 15 milliseconds (fig. 3). Then there is more time available to develop the slip surface which requires the least energy to failure, which leads to a maximum set per blow. Especially with large diameter piles the failure mechanism in the soil during driving may differ significantly from semi-static cases.

This (not exhaustive) list of influence serves to illustrate the difficulty of relating the SRD to the bearing capacity in final static or semi-static conditions. For the time being the reliability will depend heavily on the collected experience of the industry and consulting engineers in order to establish empirical correlations (ref. 2).

This is an absolute necessity because full scale load tests on large diameter piles in offshore conditions, sometimes even under water (ref. 10), are not feasible due to prohibitive costs and technical difficulties.

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**FULL SCALE DRIVING TEST IN SLIEDRECHT, THE NETHERLANDS.**

In the autumn of 1978 two HBM type 4000 Hydro- blok hammers were delivered. A duration driving test was part of the commission procedure. The objectives of the test were:

1) To prove to the client that the hammers meet the guaranteed performance specifications over an extended period of time.

2) To check hammer design calculations and to calibrate design techniques for future developments.

3) To utilize the opportunity to gather information of a more general nature about the interactive behaviour of the hammer-pile-soil system.

Part of the test related to the third objective is the subject of this paper.

During the test a 84 inch pile with a wall thickness of 1.25 inch and a closed toe was driven to a depth of 65 m. at an inclination of 6° on 1.

A maximum net impact energy of 190 tonforcem (1,370,000 ft.lbs.) was delivered at the catalog value amount to 160 tonforcem (1,160,000 ft.lbs.) (fig. 2). The buffer force was varied from its maximum value of 4000 tonforce to somewhat less than 2000 tonforce. At the final penetration the blow-count went up to over 1000 blows/ft. at full energy input.

At the time of the measurements the length of the pile was 60 m. Strain-gauge and accelerometer measurements were carried out at two levels near the top and at two levels near the toe of the pile. Apart from standard accelerometers a special strain difference technique was used to determine accelerations (see appendix).

In figure 3 the measured impact-force-time diagram is shown as well as the computed values at the same setting up of the buffer force and impact velocity. After the duration of the primary impact of about 10 ms the influence of the reflected skin friction forces and the toe resistance can be clearly discerned. It must be noted that movement of the top of the pile is restrained by the heavy mass of the ram. Comparison of computations and measurements show that properties of the hammer, the pile and the soil are adequately represented in the computation model, which is incorporated in the "PILEWAVE" pile driving analysis computer program (ref. 1).

The SRD was determined by summation of measurements of strains and accelerations and also by post-analysis of blow-counts (fig. 4). By varying the buffer force and the impact velocity at successive levels post-analysis could be performed for different impact-force time histories at different energy levels. At successive blows these should lead to approximately the same SRD. A blow-count of 42 blows per 0.25 m. for instance, corresponds with a SRD of 74 MN at an energy level of 1.90 MW and a buffer setting of 4000 tonforce. At a energy level of 1.60 MW we then expect a blow-count of 65 blows per 0.25 m. (fig. 2).

Soil investigations included CPT-tests of cone resistance and local friction (figure 5), density measurements and laboratory tests. For the CPT-tests the internationally standardized cylindrical cone with friction sleeve was used at a penetra-
tion speed of 20 mm/sec. The first 9 m of the soil profile consists of peat, then up to 40 m layers of clay and medium dense sands. From 40 m to 70 m we find dense sands and below 70 m clay with dispersed sand layers.

**ASTERN SHELDT ANCHOR PILES.**

In the Eastern Scheldt estuary 59 anchor piles were driven at waterdepths varying from 10 to 36 meters. Penetration varied from 15 to 21 meters depending on soil conditions. The 56 inch OD piles with a wall thickness varying from 24 mm at the ends to 46 mm in the middle are designed to withstand anchor forces up to 200 tonforce from barges, ships and other equipment during construction of the Eastern Scheldtstorm barrier. The piles were driven with a HBM type 1500 Hydroblock hammer which delivered a net energy of 40 tonforce meter (290,000 ft.lbs.). During driving the impact velocity and the buffer force indicated on the operators console were noted. Hence hammer data are known with sufficient accuracy to permit accurate post-analysis of observed blow-counts.

Cones Penetration Tests (fig. 7) and borings show dense sandy soils with cone resistances up to 60 N/mm² and local peaks of over 80 N/mm². Measurements indicated a ratio between local friction and cone resistance between 0.8 and 1.2%. Similar dense sand deposits are found offshore in the North Sea. Heavy driving was expected in vicinity of CPT225 and similar CPT’s which show very high cone resistances and local sleeve friction. However, maximum blow-counts per pile did not exceed 75 blows/0.25 m with an average maximum of 27 bl/0.25 m. for all piles. In figure 8 the predicted axial static bearing capacities (non-plugged condition) are shown for CPT’s 209, 225 and 237. Also are indicated 4 SRD values calculated by post-analysis for the piles R12, R16 and R23 which should be compared with predicted axial static bearing capacities based on CPT-225, -209 and -225 respectively.

**COMPARISON OF SRD BY MEASUREMENT AND POST-ANALYSIS WITH PREDICTED STATIC BEARING CAPACITY.**

The results of comparisons are presented in figure 4 for the Slindrecht test and in figure 8 for the Eastern Scheldt piles. In the latter case no direct measurements of SRD’s were performed. Post-analysis values refer to the total of skin friction and toe resistance. No damping, quake added mass and friction fatigue were taken into account. Hence the soil-pile interaction model is the same as in the post-analysis in the computation of the static bearing capacity. This constitutes a simple computation model, for which the soil parameters are based on direct measurements in the field, i.e. local cone resistance and local skin friction. If sufficient agreement is obtained, such a model is to be preferred above more complicated concepts which need assumptions on soil parameters which cannot be determined from field tests. Moreover, damping and friction-fatigue tend to balance each other; it is possible to achieve the same SRD from a post-analysis using a high damping coefficient if a large influence of friction fatigue as the found when a low damping coefficient is assumed and a moderate influence of friction fatigue is introduced.

The figure 4 shows good agreement between the SRD determined from post-analysis and the derived static bearing capacity. The SRD derived from measurements of strains leads to values which are consistently lower than the other two; results from dynamic measurements vary between 60% and 100%, with an average of 80%, of the computed static bearing capacity. Apart from deficiencies in measurement techniques this difference could be attributed to the circumstances that the SRD from dynamic measurements is a summation of forces which occur at different frequencies at different levels.

In figure 8 the results for piles R12 and R16 show acceptable to good agreement with the calculated static bearing capacity. The results for pile R23 show a considerable lower SRD obtained from post-analysis than the static bearing capacity calculated from CPT-225. Similar differences were observed in the vicinity of those other CPT’s, which also showed very high cone resistances.

This phenomenon has also been noted by other observers (ref. 13). They state that in such cases the high values of the cone resistance could be mainly determined by high horizontal pressures in the soil. These are explained by the occurrence of overconsolidation. The ratio of overconsolidation could decrease significantly during pile driving, leading to a reduction of skin friction and toe resistance.

In the case of the Eastern Scheldt a better agreement between SRD based on post-analysis of blow-counts and predicted axial static bearing capacity based on CPT is found when no higher values of the cone resistance and the local sleeve friction are taken into account in the calculation of the axial static bearing capacity than 35 N/mm² and 0.35 N/mm² respectively.

**CONCLUSIONS**

Conclusions must be restricted to soils similar to those at the Slindrecht site and the Eastern Scheldt, i.e. dense sands, and to hydraulic hammers with a pretensioned buffer resulting in approximately rectangular impact force time diagrams, which are well defined and controlled.

1) For driveability studies and calculations of SRD from post-analysis of blow-counts, a simple soil model works well. The toe resistance may be derived according to the CPT-method from Cone Penetration Tests and the skin friction can be derived from the local skin friction, provided that the maximum values of the local cone resistance of the local sleeve friction are less than 35 N/mm² and 0.35 N/mm² resp. No damping, quake added mass and friction fatigue need to be introduced. The introduction of higher values than the quoted maximums in calculations based on CPT’s tend to lead to unreliable results.

2) Careful post-analysis of observed blow-counts provide a check on the reliability of drives with respect to the axial bearing capacity.

3) The bearing capacities derived from dynamic measurements in this case tend to present a lower bound with respect to predictions based on CPT-tests and post-analysis of blow-counts.
4) Comparison of measured and calculated impact-force-time-diagrams show that the properties of hydraulic hammers, the pile and the soil are adequately represented in the computation model "PILEWAVE", that is used for post-analysis.

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The Slidrecht Pile-Test was made possible through the generous support and participation of sponsors which represent a broad international forum:

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APPENDIX

MEASUREMENT OF ACCELERATIONS BY THE STRAIN DIFFERENCE METHOD.

The described method was first suggested by C.R.G.J.S.L. Voitus van Hamme. The difference $\Delta \varepsilon$ of the normal axial strain measured at two levels in the pile is related to the acceleration $a$, according to the following formula:

$$a = \frac{\Delta \varepsilon}{\Delta t} = \frac{c^2}{h} (\varepsilon_2 - \varepsilon_1)$$

(6)

$h = \text{distance between levels 1 and 2}$
$
\varepsilon_1 \text{ and } \varepsilon_2 \text{ are the strains measured at levels 1 and 2 at the same time.}$

The method applies within the following restrictions:

1) One dimensional wave theory with a velocity $c$ must govern the phenomena.

2) Between levels 1 and 2 no external forces may act on the pile.

3) $(a \cdot \Delta t)^2 << 1$

$$f = \text{frequency of signal component and } \Delta t = h/c$$

The expression (6) follows from the equilibrium condition of the section of the pile between levels 1 and 2. The difference $\Delta \varepsilon$ between the axial forces acting at these levels must be equal to the d'Alembert inertia force.

The computed acceleration includes the gravitation effect "g". Since $a$ is an average for the section under consideration, the length $h$ of the section puts a limit on the frequencies of the acceleration that can be determined with sufficient accuracy as dictated by the expression (7). With respect to ease and speed of computation this method has been found to be superior to an alternative procedure described in reference (7) and it showed identical numerical results in practical applications.

One obvious follow-up is to construct a bar that is only connected to the pile at one end. In such a case strains measured at one section of the bar directly indicate the strain difference with respect to the free end (where strains are zero).
Problems with resonance of such a bar have been solved with simple digital filters. The result is a simple, rugged, accelerometer which has proven to be useful when measuring accelerations to obtain local velocities of steel piles.

Fig. 1 - The 84 inch OD test pile and the hydrobloc HBM 4000 on the site at Sliedrecht, The Netherlands.
Fig. 2 - Blow-count vs SRD for Hydrobloc hammer type HBM 4000 1.9 MWh = 1,370,000 ft. lbs, 1.6 MWh = 1,160,000 ft. lbs.

Fig. 3 - Impact force/time diagrams for type HBM 4000 at buffetsetting, 38 MN and impact velocity 5.5 m/s. Net energy: 1.50 MWh=1,060,000 ft.lbs.

Fig. 4 - Comparison of SRD computed from dynamic measurements and from post-analysis of blow-counts vs static bearing capacity computed from CPT's Biedrecht.
Fig. 5 - Cone resistance and local friction obtained from cone penetration at Sliedrecht site.

Fig. 6 - Driving of anchor piles in the Eastern Scheldt Estuary with the Hydromet HEM 1500.

Fig. 7 - Cone resistance from cone penetration tests, Eastern Scheldt.

Fig. 8 - Comparison of SBD computed from post-analysis of blow-coring data vs static bearing capacity computed from CPT's, Eastern Scheldt.
ESTIMATING FRICTION PILE LENGTHS FROM BORING DATA

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Howard Needles Tammen & Bergendoff, New York, NY 10019

For presentation at
Associated Pile & Fitting Corp. PILETALK Seminar
Miami Beach, Florida, March 1978

A. Seymour-Jones is chief soils engineer for HNTB's eastern offices. He is in charge of field laboratory, and design work related to subsurface materials and structural support. He has had extensive experience in hydraulics, geology and foundations engineering. Recently he has worked on evaluation of support in potential slide and coal mine waste areas, on Washington DC Metro, and on a major highway interchange with a bridge over a former mine. Other work has included waterfront structures such as cellular cofferdams, airfield planning and underpinning of buildings.

A difficult task for engineers and contractors is estimating the lengths of friction piles. A theoretical equation has not been developed that results in accurate pile length estimates. Empirical methods, rules of thumb and judgment based on experience are used.

This paper presents analytical guides for estimating the lengths of friction piles that the author has found useful. There is no claim of originality for these guides since they were developed from methods proposed by others, which the author has found to be useful. The use of these guides requires a critical review of the results obtained to ensure that they are reasonable.

Estimating length of friction piles is used primarily by foundation engineers to assure that adequate pile-friction capacity is achieved and by engineers and contractors to estimate pile contract quantities.

The guides presented herein meet both of these needs. The writer judges these guides to be applicable for driven piles up to 150 ton design load.

The basic data required to utilize these guides are boring logs containing the Standard Penetration Test (SPT) data, soil descriptions, and information on the proposed type of piling to be used such as type, design load and shape.

It is not the intent of the author to imply that these guides should replace pile load tests. Rather, they can be used as a means to estimate the length of load test piles, to supplement the load test results where soil conditions are quite variable and to provide an estimate of pile quantities prior to the making of pile load tests.

These guides are not intended to be used to evaluate potential pile settlement or pile group effects. Additional studies, which are beyond the scope of this paper, are required for such evaluations.

Boring Data

The basic data required from the borings is the SPT spoon penetration resistance and a reasonably good soil description. The use of these guides requires reasonably accurate boring data.

It is not the intent to imply that soil strength tests or special field tests will not be required. Often these guides can provide insight as to when such
are available or when the project is too small to warrant special testing.

It has been established that the SPT data can be approximately related to the cohesion of clay soils (Terzaghi & Peck) and to the effective density of granular soils, which can be related to the angle of shearing resistance for granular soils (Burmister). These factors provide the basis for using the boring data.

Pile Data

At the present time there is a wide variety of driven piling in use. Characteristics inherent to different types of piles affect their capacity and driving and must be taken into account when using these guides. The basic information required for this analysis is whether the pile is a non-displacement, straight displacement or a tapered displacement pile.

Static Pile Capacity Method

A number of writers, such as Moore, Chellis, Terzaghi and Peck, have developed analytical methods for determining the capacity of friction piles. Additional methods are presented in the Navy Design Manual DM-7. The basic analysis inherent to all these methods is the estimation of the pile-soil adhesion values developed over the full length of the pile. Usually some estimate of the effect of the pile point bearing is also included.

The writer's method of analysis uses the same approach. The soil friction values are determined as a function of the Standard Penetration Test values, the soil description and the type and shape of the proposed pile. A chart relating these effects was developed, based primarily upon the pile-soil adhesion values noted by Chellis and Terzaghi and Peck. This method neglects any point bearing contribution to the pile capacity since tests on instrumented piles have shown that it generally contributes 20 percent or less of the pile capacity. Fig. 1 presents estimated values of pile-soil adhesion values based upon the conditions listed above.

The use of this method of analysis is simple. After the desired pile type, working load and safety factor are determined, the incremental values of pile-soil shear are determined for the pile until a depth of pile equaling the required pile capacity is determined. A sample calculation is given in Appendix I. It should be noted that the above method of analysis requires separate evaluations of the pile capacity as a structural member, the pile group effects and pile settlement.

The writer has compared this method of analysis with the results of pile load tests where the soil conditions have been relatively uniform. The results are given in Fig. 2. The data indicates this analysis is generally conservative, probably at least partially due to omission of the bearing capacity of the pile point.

Dynamic Pile Formula Method

The Bureau of Reclamation has developed a method of estimating the length of friction piles based on a comparison of the SPT spoon sampler driving resistance with the driving resistance on a number of timber piles analyzed by the Engineering News formula. This approach was modified by a former associate of the writer, Guy Tabor, and is presented here.

The EN formula is known to have poor accuracy in estimating the bearing capacity of friction piles; a great number of people have recommended that
Employ it during pile driving to determine the required capacity. Consequently, it is reasonable to use the above noted method to estimate pile lengths for projects where pile driving will be controlled by the EN formula. This method of analysis has been found to provide good estimates of friction pile lengths on projects where pile driving was controlled by the EN formula.

**Engineering News formula**

For drop hammers

\[ R = \frac{2 \ W \ H}{S + 1} \]

For single-acting hammers

\[ R = \frac{2 \ W \ H}{S + 0.1} \]

For double-acting hammers

\[ R = \frac{2E}{S + 0.1} \]

where

- \( R \) is the allowable pile load in pounds
- \( W \) is the weight of striking part of hammer in pounds
- \( H \) is the effective height of fall in feet
- \( E \) is the actual energy delivered by hammer per blow in ft-lbs.
- \( S \) is the average net penetration in inches per blow for the last five blows after the pile has been driven to a depth where successive blows produce approximately equal net penetration.

The approach is to relate the SPT spoon sample driving resistance to the pile design load by empirical constants that are a function of the pile type, shape and soil type. The basic formula is:

\[ P = \sum NLf \]

where

- \( P \) is the design pile load in tons
- \( N \) is the Standard Penetration Resistance in blows per foot
- \( L \) is the depth of soil represented by \( N \) in feet
- \( f \) is the empirical constant

Fig. 3 gives the values for “f” and notes the appropriate pile types and soil conditions to which they apply.

It has been found that the maximum \( N \) value that should normally be used in this equation is 50.

When SPT boring data is presented from 5 ft intervals and soil conditions are relatively uniform this equation can be readily used in a simplified form

\[ \sum N = \frac{P}{5f} \]

to calculate the expected pile length as illustrated in the example in Appendix I. Computations are for pipe piles. The same figures are applicable to uniform diameter corrugated shell. For H-piles the square size of the pile is used.

There are a number of restrictions concerning the use of this method of analysis that should be noted. (1) It should only be used for projects where a dynamic pile driving formula is to be used as the basis for driving the piling.
FIGURE 3

Values of Constant "f"

Note:
- * denotes clay soils
- 0 denotes sand soils

(2) It has been found that in fine sand deposits the sand may develop a quick
condition and lose resistance under the driving of the pile, resulting in the
piles penetrating significantly deeper than the length estimated by this
method of analysis. (3) If the piles are to be pre-augered then this method
can be expected to significantly under estimate pile lengths. (4) The data
noted for large diameter cylinder piles is tentative because it is based on very
limited data. It should be used with caution.

When using this method of analysis the results should be checked against
the boring logs to insure that they appear reasonable. Empirical formulas, of
which this method is one, are only guides, which should be incorporated with
good engineering judgment.

General Comments

It has been the author's experience that the use of dynamic pile driving
formulas for estimating the length of friction piles for some stiff clay deposits
results in piles much longer than required. For these cases the piles should be
driven to a specified pile tip elevation determined from a pile load test. The
determination of the optimum tip elevation for the load test pile should be
based on the or some other static pile capacity method.

When estimating the length of friction piles using the two guides presented
above it is best to determine an expected range in pile lengths. Once the
range of values has been determined a final single value can be selected by
evaluating the limiting values of the range against the boring log.

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APPENDIX I

ESTIMATED PILE LENGTHS - EXAMPLES

1. Static Pile Capacity Method

Design Criteria: 35T AASHO Group I Load, Safety Factor = 2, 12" d pipe pile

Formula

\[ L = \frac{P \times F_s}{A_c \Delta A} \]

where:
- \( P \) = pile design load = 35T - 70k
- \( F_s \) = Safety Factor = 2
- \( A_c \) = Pipe circumference/area/ft
- \( \Delta A \) = Length of pipe segment
- \( \alpha \) = Pipe to soil adhesion for \( \Delta A \)

\[ L = \frac{70 \times 2}{3.14} = 44.5 \]

\[ \Delta A = \frac{70 \times 2}{3.14} = 44.5 \]

<table>
<thead>
<tr>
<th>Soil Depth</th>
<th>Description</th>
<th>N</th>
<th>( \Delta L )</th>
<th>( N_a )</th>
<th>( \alpha ) kips</th>
<th>( \alpha \times \Delta L )</th>
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<tbody>
<tr>
<td>0</td>
<td>Sandy</td>
<td>5</td>
<td>Ftg. level</td>
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<td>1.1-1.4</td>
<td>5.5-7.0</td>
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<td>10</td>
<td>Sandy</td>
<td>10</td>
<td>5'</td>
<td>10</td>
<td>0.4-0.6</td>
<td>2.8-4.2</td>
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<td>20</td>
<td>Silt</td>
<td>8</td>
<td>7'</td>
<td>8</td>
<td>1.2-1.6</td>
<td>21.6-28.8</td>
</tr>
<tr>
<td>30</td>
<td>Sand</td>
<td>12</td>
<td>18'</td>
<td>12</td>
<td>0.9-1.3</td>
<td>4.5-5.6</td>
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<tr>
<td>40</td>
<td>Clay</td>
<td>16</td>
<td>5'</td>
<td>16</td>
<td>5.5-8.0</td>
<td>39.9-54.5</td>
</tr>
<tr>
<td>50</td>
<td>Sand</td>
<td>21</td>
<td>5'</td>
<td>21</td>
<td>1.1-1.6</td>
<td>5.5-8.0</td>
</tr>
<tr>
<td>60</td>
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<td>17</td>
<td>7'</td>
<td>17</td>
<td>0.7-0.9</td>
<td>4.9-6.3</td>
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<tr>
<td>70</td>
<td></td>
<td>22</td>
<td>Use 44' (L=39')</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Dynam. Formula Method

Design Criteria: 35T working load
(a) check 12" pipe pile
(b) check tapered pile

(a) use f=0.055 to 0.060, for△L=5'; $N = \frac{35}{5 \times 0.055} = 117$ to 127

(b) use f=0.070 to 0.080, fo△L=5'; $N = \frac{35}{5 \times 0.07} = 88$ to 100

Boring Data

<table>
<thead>
<tr>
<th>Depth</th>
<th>Soil Description</th>
<th>N</th>
<th>ΣN</th>
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<td>0</td>
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<td>Sandy Silt</td>
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<td>45</td>
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<td>50</td>
<td>Clay</td>
<td>16</td>
<td>73</td>
</tr>
<tr>
<td>40</td>
<td>Sandy Sand</td>
<td>29</td>
<td>102</td>
</tr>
</tbody>
</table>
| 43    | Layers Sand & Clay | 18 | 145 (a) Est. pile tip at 42' to 43' Use 43' (L=38')
| 60    | Layers Sand & Clay | 22 | 15  |
| 70    | Layers Sand & Clay | 15 | 17  |
STRESS HISTORY APPROACH TO ANALYSIS OF SOIL RESISTANCE TO PILE DRIVING

by Robert M. Semple and J. Peter Gemeinhardt, Mcclelland Engineers

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ABSTRACT

A method has been developed to predict soil resistance during continuous pile driving in clay base. On interpretation of pile installation experience and related soils data in terms of the soil overconsolidation ratio. Soil resistance, obtained from offshore pile driving records using standard wave equation analysis, can be less than or greater than the predicted static pile capacity in normally consolidated and heavily overconsolidated clays, respectively. Analysis of installation data from large diameter pipe piles in soft to hard clays provides an empirical factor for computing soil resistance from the predicted static pile capacity. Values of the factor for six case histories show a consistent relationship with overconsolidation ratio, and this relationship is represented mathematically. The procedure incorporates overconsolidation ratios determined from undrained shear strength and soil index properties. The method for computing soil resistance during driving for a given soil profile based on the static pile capacity and the empirical adjustment factor is described.

INTRODUCTION

Drivability of offshore pipe piles is currently predicted by estimating the resistance that a pile-hammer combination can overcome and comparing this with the likely resistance of the foundation soils. The capability of pile-hammer combinations to overcome resistance can be analysed using one-dimensional stress wave theory (Smith, 1962). A logical starting point for assessing the soil resistance is the predicted static axial compressive pile capacity. However, pile installation data indicate that soil resistance during continuous pile driving is less than the predicted static pile capacity in normally consolidated clays (McClelland et al, 1969; Aurora, 1980; Stockard, 1980), and can exceed the static capacity in heavily overconsolidated clays (McClelland et al, 1969; Fox et al, 1970; Durning and Rennie, 1978). In order to examine the relationship between static pile capacity, the soil resistance mobilised during driving, and the degree of overconsolidation, six installation case histories of large diameter pipe piles in silty and sandy clays were analysed. The results indicate that the overconsolidation ratio of cohesive soils has a significant influence on pile drivability. The paper presents results of this study and a procedure for computing soil resistance during continuous pile driving.

References and Illustrations at end of paper

STRESS WAVE THEORY

Stress wave theory (Smith, 1962) has been used routinely for many years in offshore pile drivability studies. Nonetheless, uncertainty exists regarding values for some of the numerous parameters used in the mathematical solution. The more important of these variables are the hammer impact energy and the parameters for springs and dashpots that model soil resistance. Hammer input energy is not routinely measured in offshore construction. Conflicting published values for quake and damping, representing the resistance springs and dashpots respectively, indicate an imperfect knowledge of appropriate values for soils. Application of a reasonable range of values for quake and damping for a given soil results in variation by a factor of at least two in the predicted resistance that a given pile-hammer combination can overcome. Variation of hammer efficiency increases the range of predicted resistance.

Considering the uncertainties regarding key wave equation variables, the most reasonable approach at present is to use the wave equation in a standardised manner to avoid introducing additional uncertainties into pile drivability analyses. For this study of piles in clay, quake was taken as 2.5 mm, and side and tip damping as 0.65 and 0.15 sec/m, respectively. Side resistance was assumed to increase linearly from the seafloor. The proportion of tip resistance was taken directly from static pile capacity calculations. Efficiency was taken as 75 percent for the larger offshore steam hammers considered herein except where measurements indicated actual values. This combination of values for the parameters of primary importance produces wave equation results that are representative of normal practice in the offshore industry today.

Where comments are made about results of wave equation analyses in this paper, the foregoing parameters are applied. In reality, the magnitude of soil resistance corresponding to a given rate of penetration is not accurately known because of uncertainty in application of the wave equation model.

PILE CAPACITY

Shaft capacity of offshore pipe piles in clay usually is computed by either the methods given in API RP 2A (1980).
or the "\(\lambda\)-method" proposed by Vijayvergiya and Focht (1972). For clays of low to medium plasticity, API recommends values that vary with the soil undrained shear strength. Both the \(a\) and \(k\) values are based on pile load test data assembled and presented by Vijayvergiya and Focht (1972). These data are plotted on Fig. 1 as skin friction versus undrained shear strength. Published data for hard, overconsolidated, lean clay are limited to tests on two 0.76 m (30 in.) diameter conductors performed at BP's West Sole Field in the North Sea (Fox et al, 1970, 1976). In this study, the API relationship, the solid curve shown on Fig. 1, was used for undrained shear strengths up to 400 kPa. Skin friction was taken as 200 kPa for stronger overconsolidated, lean clays as linear extrapolation of the empirical relationship for such soils is uncertain. Limiting adhesion does not apply to normally consolidated clays that have high undrained shear strengths because of existing overburden stress.

Unit end bearing was computed as nine times the undrained shear strength of the clay at pile tip penetration. However, total end bearing was limited by the frictional resistance of the soil plug. The behaviour of a soil plug is complex (Poskitt, 1978), and is likely to be different under dynamic and static loading. Many pipe piles are installed with an internal driving shoe to reduce internal skin friction. Comparative data of piles in clay with and without a shoe indicate that an internal shoe can reduce the driving resistance and the extent to which the pile plugs during driving (Heerema, 1979; Fox et al, 1976). Opinions vary regarding the reduction in internal skin friction caused by a shoe during continuous driving. Some believe that an internal shoe completely eliminates internal skin friction in stiff clay, whereas others assume reductions of 30 to 50 percent (Toolan and Fox, 1977; Durning and Rennie, 1978; Heerema, 1979). For assessing pile drivability, internal skin friction was assumed equal to the external skin friction for flat-end piles, and 25 percent of external skin friction for piles with an internal driving shoe. The internal clearance will not be maintained after pile installation so the effects of a shoe on driving resistance and on long term pile capacity are not the same.

**SOIL BEHAVIOUR DURING DRIVING**

Considerable driving experience is available for long pipe piles in the Gulf of Mexico where cohesive soils are predominately under-to-normally consolidated and highly plastic. Soil resistance during driving computed using driving records and standard wave equation analysis is generally 20 to 50 percent of the relevant static capacity (Aurora, 1980; Stockard, 1980). The reduction in resistance is attributed to remoulding and loss of shear strength in these relatively sensitive clays. Remoulded undrained shear strength has been used to predict soil resistance in stiff overconsolidated clays (Toolan and Fox, 1977).

In stiff clays, however, the driving resistance can increase with penetration to 20 or 30 m depth then remain unchanged as the pile is driven to greater depths whereas the computed static capacity increases more or less linearly from the seafloor. A good example is the well-documented installation of Union Oil's Heather platform in the North Sea (Durning and Rennie, 1978; Durning et al, 1978). One explanation for this behaviour assumes that the unit skin friction at a given penetration progressively decreases with the length of pile driven past that depth. Heerema (1978) developed a procedure to account for this possibility in addition of soil resistance based on the Heather platform experience and data from several other sites. An alternative explanation of the Heather driving resistance is based on the overconsolidation ratio (OCR) of the cohesive soils. At shallow depths the Heather clays are very heavily overconsolidated, but the OCR reduces rapidly with penetration, the soils becoming near normally consolidated at final pile penetration. If soil resistance during driving is related to OCR then the type of driving resistance observed at Heather Field, i.e. generally increasing with depth at a decreasing rate, is to be expected.

**Stress history, as expressed by OCR, significantly affects soil behaviour.** Two clays with the same undrained shear strength can have quite different stress histories which are reflected in their stress-strain and volume or pore water pressure change characteristics during shear. In undrained shear, normally consolidated clays tend to strain soften, their particle structure collapses and the tendency to volume decrease produces relatively large positive pore water pressures. Heavily overconsolidated clays tend to strain harden with dilatant behaviour during shear resulting in negative pore pressures. Pile installation shears clay to large strains, so stress-strain characteristics, and hence stress history, should affect resistance to pile driving. A parallel exists with regard to static skin friction. The OCR values for cohesive soils should be related to OCR rather than undrained shear strength (Wroth, 1972; McClelland, 1974), and this relationship can be seen in pile load test results (Semple, 1979).

The relationship between static pile capacity, soil resistance during driving and OCR was elucidated in general terms more than a decade ago by McClelland et al (1969). Soil resistance is significantly less than static capacity in normally consolidated clays and exceeds it in heavily overconsolidated clays. Where the OCR is initially large and decreases to small values at depth, as at Heather Field, the soil resistance may initially exceed the static pile capacity and then reduce below it at depth. These trends indicate the possibility of determining from pile driving experience a factor, varying with OCR, that can be applied to the computed static pile capacity to predict soil resistance during driving.

**PILE CAPACITY FACTOR**

The ratio of computed static pile capacity, \(Q\), to soil resistance during continuous driving, \(R\), was defined by Aurora (1980) as the driving resistance factor, \(F_D\). Values of \(F_D\) were computed for some Gulf of Mexico clays at penetrations where pile driving stopped. The factor relating static pile capacity and soil resistance during driving in this study, termed the pile capacity factor \(F_P\), differs from the driving resistance factor \(F_D\) in three ways: (1) \(F_P\) relates to incremental shaft friction, that is \(BD/BR\), (2) pile capacity for lean, overconsolidated clays is computed with a limit on adhesion, and (3) end bearing is modified for a pile with a driving shoe. These variations to the conventional API method were discussed earlier. For a long pile in normally consolidated clay, the OCR is constant, the adhesion limit is not applicable, and end bearing is not significant; so the factors \(F_P\) and \(F_D\) will be approximately the same.

**EVALUATING OCR**

Values of OCR can be determined directly from laboratory one-dimensional consolidation test data, and estimated from generalised relationships between consolidation pressure and liquidity index (Nayarambath et al, 1979) or between OCR and undrained shear strength (Ladd and Foot, 1974; Kourvoias and Fischer, 1976; Ancersen et al, 1979). Laboratory consolidation test data usually are not available for published pile installation records; they also tend to be affected by sample disturbance and exhibit
measurement scatter thereby introducing another element of subjective interpretation into a procedure based on OCR. Of the two generalised methods, the correlation between OCR and undrained shear strength is more precise, although the implication of any generalised relationship to specific sites can always be questioned. However, in back analysing field experience to develop a predictive technique, the absolute accuracy of a component, such as OCR, may not be of great importance provided one method is used, consistently throughout.

Relatively high quality data from simple shear tests relating OCR and undrained shear strength of lean clays are assembled on Fig. 2. Undrained shear strength, \( S_u \), is normalised to the value for the clays in the normally consolidated state, \( S_{unc} \). By normalising the shear strength data, the information can be reasonably extended to compression test data, the basis of the API method for determining shaft friction. The curve drawn through the data in Fig. 2 is described by the equation:

\[
\frac{S_u}{S_{unc}} = OCR^{0.85}
\]

The curve has been extrapolated to an OCR of 100. Undrained shear strength of normally consolidated clay may be computed from (Skempton, 1957):

\[
c/p = 0.11 + 0.0037PI \]

where \( c \) is \( S_{unc} \), \( p \) is the effective overburden stress, and \( PI \) is the plasticity index of the clay. When plasticity data are unavailable, the \( c/p \) ratio for a lean clay may be taken as 0.2 and the submerged unit weight of a stiff clay as 10 kN/m\(^3\). Hence, \( S_{unc} \) in kPa units is twice the depth in metres. For a stiff, lean clay the OCR may be estimated as:

\[
OCR = 0.45 \left( \frac{S_u}{2} \right)^{1.2}
\]

where \( S_u \) is the undrained shear strength (kPa) at depth \( Z \) (m) taken from the shear strength profile established for static pile design. In kip, \( Z \) units the constant of proportionality in Eq. (3) is 200. When submerged unit weight and plasticity data are available, Eq. (2) should be evaluated directly.

PILE INSTALLATION INFORMATION

Extensive information on pile driving in under-normally consolidate, highly plastic clay is available from the Gulf of Mexico and is represented in recent publications by Aurora (1980) and Stockard (1980). Two major platform installations featuring cohesive soils in the North Sea have been well documented: British Petroleum's Forties platforms (Hirsch et al., 1975; Fox et al., 1976; Sutton et al., 1979) and Union Oil's Heather platform (Durning and Rennie, 1978; Durning et al., 1978; Raasche, 1978). Texaco made available information for their Tartan platform.

Heather Field is at the northern end of the North Sea, about 100 km east of the northern tip of the Shetland Islands. Forties and Tartan Fields lie about 80 km apart, 300 to 350 km south of Heather and about 150 km east to northeast of Peterhead, Scotland.

The Heather platform was installed in about 140 m of water in Block 2/5 of the UK North Sea in the summer of 1977. The platform is supported on four groups of six 1.52 m (60 in.) diameter by 64 mm (2.5 in.) wall piles fitted with an internal shoe and driven to about 46 m penetration primarily by the Menck 8000 and 12500 steam hammers. The four

Forties platforms are located in Block 21/10 of the UK North Sea where water depths range from about 110 to 130 m. Groups of 9 and 11 plain-end 1.37 m (54 in.) diameter by 51 to 64 mm (2 to 2.5 in.) wall piles were driven to a maximum penetration of 76 m. At the FD platform, an additional instrumented pile, with observations of hammer performance, was installed using the Menck 3000 and 7000 hammers in the spring of 1976. The Tartan platform was installed in the summer of 1979 in 180 m water depth in Block 15/16 of the UK North Sea. The platform is supported by groups of seven 1.83 m (72 in.) diameter by 64 mm (2.5 in.) wall plain-end piles driven to about 67 m penetration using the Menck 8000 hammer.

Geotechnical parameters for the silty and sandy clays at the Heather, Forties FD and Tartan platform sites of primary relevance to this study are shown on Fig. 3. Foundation conditions at these sites are fairly representative of relatively strong and weak clays in the North Sea. Based on published data (Holmes, 1977), the shallow soils at Forties and Tartan appear to have the same geological origin. At Tartan, an intermittent stratum of dense silty fine sand was present at the base of the normally consolidated clay. The \( S_u \) profiles for the three platforms were developed for pile design at the time of the investigation. The \( S_{unc} \) and OCR profiles were obtained by applying Eq. (1) to (3). The OCR values for all three locations are shown on Fig. 4. Data from three other platform sites available in our files also were analysed. The OCR profiles and \( S_u \) values for these additional sites fall within the range represented by Heather, Forties and Tartan.

Soil resistance during continuous driving interpreted using wave equation analyses from the field penetration rate data are presented on Fig. 5. The Heather data were taken directly from the paper on the platform installation (Durning et al., 1978). The published driving record for the instrumented pile at Forties FD platform (Sutton et al., 1979) was interpreted for this study using a hammer efficiency of 65 percent based on the reported observations of the hammer performance. The range of soil resistance at Tartan was affected by the intermittent sand stratum that was not present at all pile positions. Undrained shear strength in the deeper clays at Tartan also varied locally, tending to be greatest at places where the sand stratum was encountered. The lower and upper bound soil resistances are averages for the four legs of piles that respectively appeared to miss and intersect the sand stratum. Similar data were analysed for three other platforms supported by large diameter pile pipes installed with steam hammers.

DATA ANALYSIS

Static axial pile capacities at various penetrations below the seafloor were computed for the six platforms using the criteria outlined previously. Static capacity curves for the Heather, Forties and Tartan soil conditions are shown on Fig. 5. As a prediction of soil resistance during driving, the static capacities are of the correct magnitude, but some differences exist that would significantly affect pile installation planning and performance. The maximum soil resistance at Heather exceeds the maximum computed static pile capacity by some 20 percent, whereas the static capacity at Forties is about 70 percent greater than the soil resistance at terminal penetration.
Factors required to adjust the static pile capacity to fit the maximum soil resistances were determined by comparing the change in static shaft capacity, \( \Delta Q_s \), with the estimated change in soil shaft resistance during driving, \( \Delta R_s \), for successive increments of depth, and then computing the pile capacity factors \( F_p \) as \( \Delta Q_s / \Delta R_s \). The static capacities and soil resistances were reduced by the end bearing component prior to analysis. Unit end bearing was assumed to be the same.

Incremental analysis of two variables introduces problems of scatter in the results. Back analysis of the \( F_p \) values is sensitive to details of static capacity and soil resistance variations with depth, and these can vary significantly over successive depth increments. To overcome this problem, the curves of soil resistance during driving were smoothed, which is unavoidably subjective. The validity of the analyses so performed can only be judged by the usefulness of the results. However, just as incremental back analysis induces scatter in the resulting data, forward prediction based on these data tends to integrate variances so that the magnitude of scatter is to some degree misleading. In other words, a pile tends to integrate the response of a soil profile. Predicting the maximum value of soil resistance and not the details of a soil resistance profile is most significant for pile hammer selection.

In dividing pile penetrations into depth intervals for analysis, a constant depth increment of 3 m was used wherever possible so that all resulting \( F_p \) data could be given equal weight. For ease of computation, the actual depth increments varied from about 2.5 to 3.5 m. Because of the influence of the sand stratum on driving resistance at Tartan, analyses were made only for increments in the stronger clay below 43 m penetration. A total of 90 individual \( F_p \) values were computed for the six platforms.

The \( F_p \) values were plotted against \( S_{u1} \), depth, and OCR for the various depth increments. No relationship with \( S_{u1} \) was apparent. \( F_p \) tended to decrease with increasing pile penetration, less scatter in the six sites indicated a different relationship. When depth was replaced by OCR, the results collapsed into a well-defined zone as shown on Fig. 6. Most \( F_p \) values are between 0.5 and 2. A relationship representative of the data is:

\[
F_p = 0.5 \text{(OCR)}^{0.35} \quad \ldots \quad (4)
\]

The relationship was selected to provide a value of 0.5 for a normally consolidated clay in recognition of the considerable experience from the Gulf of Mexico that indicates soil resistance during continuous driving generally does not exceed one-half of the computed static pile capacity. A clay profile having an average OCR of 10 is predicted to have a soil resistance during continuous driving equal to the computed static pile capacity.

**APPLICATION OF PROCEDURE**

Soil resistances obtained from computed static pile capacities modified for OCR by use of Eq. (4) are shown for the Heather, Forties FE and Tartan platforms on Fig. 5. Soil resistance in the sand stratum at Tartan was computed using a unit skin friction of 100 kPa, and a unit end bearing of 30 MPa acting on the steel annulus of an unplugged pile. The pile capacity factor method overestimates the maximum soil resistance at the three platform sites by 5 to 10 percent.

Application of this procedure involves no additional soil data to that routinely developed for basic foundation design of a piled structure. The soil parameters required are submerged unit weight, plasticity index and undrained shear strength. The first two parameters are used to compute the undrained shear strength of the soil in the normally consolidated state based on Skempton's c'/p ratio, Eq. (2). The ratio of the actual to normally consolidated values of undrained shear strength then are used with Eq. (1) to develop an OCR profile for the site. This in turn permits evaluation by Eq. (4) of the appropriate \( F_p \) values. Alternatively, Eqs. (1) and (4) can be combined to give the factor directly in terms of undrained shear strengths:

\[
F_p = 0.5 \left( \frac{S_u}{S_{unc}} \right)^{0.35} \quad \ldots \quad (5)
\]

Specifically for stiff, lean clays Eq. (5) can be approximated as:

\[
F_p = 0.4 \left( \frac{S_u}{Z} \right)^{0.35} \quad \ldots \quad (6)
\]

where \( S_u \) is in kPa and \( Z \) is in metres. In kip, ft units the constant of proportionality in Eq. (6) is 2.4. Where site-specific data are available, the \( F_p \) values for a given soil profile should be based on application of Eq. (1), (2) and (4).

To use this procedure, computations are performed to determine cumulative skin friction versus depth for the site. In overconsolidated clays, unit skin friction is limited to 200 kPa. The pile capacity factors are incorporated into these computations by multiplying each increment of skin friction by the \( F_p \) value appropriate to the average OCR for that depth increment. End bearing is then added to the adjusted cumulative skin friction. For a pile with a driving shoe, internal skin friction is taken as 23 percent of the outside friction in determining end bearing. The resulting capacity curve is the prediction of soil resistance during driving. As the method is based on the conventional undrained shear stress profile developed for the site, design axial pile capacity and soil resistance during driving are directly linked. Once the design shear stress profile is established, drivability predictions can proceed without the need to develop additional soil data that inevitably require additional interpretive judgements.

Delays once the driving sequence is initiated can lead to a further increase in soil resistance. Some data on set-up effects are available in the literature (Housel, 1970; Seed and Reese, 1957; Eide et al, 1961; Sternmac et al, 1969; Flaate, 1972; McClelland and Lipscomb, 1972; Fox et al, 1976; Aurora, 1980).

**ACKNOWLEDGEMENTS**

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**REFERENCES**


Fig. 1 - Skin Friction in Clay from Pile Load Test Data.

Fig. 2 - Effect of Overconsolidation on Undrained Shear Strength.
Fig. 3A - Geotechnical data of North Sea platform site.

Fig. 3B - Geotechnical data of North Sea platform site.

Fig. 3C - Geotechnical data of North Sea platform site.

Fig. 4 - Range of OCR conditions analysed.
**Fig. 5A - OBSERVED AND COMPUTED SOIL RESISTANCES.**

**Fig. 5B - OBSERVED AND COMPUTED SOIL RESISTANCES.**

**Fig. 5C - OBSERVED AND COMPUTED SOIL RESISTANCES.**

**Fig. 6 - PILE CAPACITY ADJUSTMENT FACTOR VERSUS OCR.**
DYNAMIC PREDICTION OF PILE STATIC BEARING CAPACITY

By Robert H. Scanlan, M. ASCE, and John J. Tomko

INTRODUCTION

This paper undertakes, through use of a continuous elastic theoretical pile model, to predict the static bearing capacity of example piles from dynamic measurements taken while driving the piles. The field measurements referred to, made on both reduced-scale and full-scale piles, consist of the force and acceleration measured as functions of time at the top of the piles during driving. The prediction scheme employs a four-parameter model of elastic and rigid-body pile response to the measured hammer force input. When this scheme is employed to match analytically the time-varying pile velocity and displacement derived from the acceleration measurements, it then also yields an estimate of the pile ultimate static bearing capacity valid just after driving. This bearing capacity is verified by direct comparison with field static tests. For cases where a "set-up" time after initial driving has occurred, most of the method reveals the change in bearing capacity realized by the pile. For this, at least one blow of re-driving is required. Finally, a simplified approximation to the given scheme is presented for engineering use, suggesting, among other things, that a routine dynamic test may be employed to determine pile static bearing capacity.

While many authors have employed the wave equation in relation to pile driving, the present study does not borrow directly from their methods. A broad search through the literature, (1), covers a number of these papers however. Important contributors in this area have been Smith (2,3), Forehand and Reese (4), and Samson, Ilyash, and Lowery (5,6,7).

Note. — Discussion open until August 1, 1969. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Soil Mechanics and Foundations Division, Proceedings of the American Society of Civil Engineers, Vol. 95, No. 3, March, 1969. Manuscript was submitted for review for possible publication on March 4, 1968.

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3 Numerals in parentheses refer to corresponding items in Appendix VI.
THE THEORETICAL PILE MODEL

Consider the pile model of length \( L \) shown in Fig. 1, where \( X \) represents the rigid body motion of the pile, \( x \) the distance coordinate along the pile of a system of axes moving with the pile and \( \xi \) the elastic deflection of the point \( x \) away from its rest position. Let concentrated forces, functions of time, \( F_T(t) \) and \( F_B(t) \), act upon the top and bottom respectively of the pile, and let \( R_L(x,t) \) be the distributed resistive force of the soil along the length of the pile.

The top force \( F_T(t) \) is the hammer force transmitted to the pile. \( F_B(t) \) is a concentrated reactive force due to the soil resistance; \( R_L(x,t) \) is a distributed frictional force. In the model to be considered here, \( F_T(t) \) will be taken as the actual input force of the hammer as measured at the pile top for example, by a suitable force transducer. (See Refs. 8 and 9 and Appendix II.)

\[ \frac{F_T(t)}{x=0} \]

\[ \frac{F_B(t)}{x=L} \]

FIG. 1.—THEORETICAL MODEL OF PILE WITH DRIVING AND RESISTIVE FORCES

\( F_B(t) \) will be taken as zero in the present study, its effect (if any) being included in the distributed resistive force \( R_L(x,t) \), which will be discussed subsequently.

The basic governing equation for the pile is the elastic wave equation. Details of this are considered in Appendix I. At the pile top \( (x = 0) \) the total dynamic displacement under the hammer blow is expressed by \( Z(0,t) \)

\[ Z(0,t) = X + \xi \]  \hspace{1cm} (1)

in which \( Z = X + \xi \) is the sum of rigid body \( (X) \) and elastic \( (\xi) \) contributions. The net result (solution of the wave equation) may be stated in the form

\[ Z(0,t) = \frac{X_c(t)}{L} + \frac{2}{L} \sum_{n=1}^\infty I_c(n, t) \]  \hspace{1cm} (2)

FIG. 2.—DRIVING (HAMMER) FORCES ON TEST PILES

Comments on the form assumed for these laws are also made in Appendix I. Essentially, of the examples treated in the present work, the laws are taken as (a) a simple distributed Coulomb resistance along the sides and (b) \( F_B = 0 \), i.e., no point bearing.

In calculation, the Coulomb resistance was taken to be very small until the time \( t_i \), corresponding closely to the peak of the measured velocity response curve; the full resistance value was employed thereafter. This full resistance value is called \( C_n \) and the determination of a proper value for it is the prin-
The damping ratio $\xi$ was taken as large; $\xi = 0.9$ and was varied very little from this value in all calculations. The time $t$ was taken near the value $C/\omega$, though not exactly at this value, but in the case of a full-scale pile at a value close to the time of the maximum measured velocity response. The measured value for the speed of sound in steel which was used was $c = 17200$ fps. For better match of predicted and measured velocity response it was found convenient to vary $t$ slightly.

The computer-aided procedure then used was as follows: given the measured force input and the theory developed in Appendix I, the best possible match between the theoretical and measured pile displacement was sought using any reasonable values for the four parameters mentioned. The measured displacement was obtained through double integration of the measured acceleration. Often an intermediate aid in determining a good displacement match was to bring computed velocity peaks into agreement with experimental ones. When a "best" match was judged to be obtained, the corresponding value

**Analysis Procedure**

Four parameters were chosen as variable for regulating the analytical pile model. These are (see Appendix I): $C_0$, the earth resistance along the sides of the pile; $K_0$, a soil shear resistance parameter; $\xi$, the damping ratio of longitudinal elastic modes in the pile under the assumed resistance law; and $t_0$, nominally the time for pile-top velocity to reach its maximum. Of these parameters, $C_0$ proved by far the most important, the others being "trimming" parameters only. (Choosing $K_0 = 0$, $\xi = 0.9$, $t_0 = \frac{C}{\omega}$ for alternately $t_0 = \text{time to reach peak velocity}$ would effectively remove these trimming parameters from further consideration.) For example, almost any reasonable value of $K_0$...
COMPARISON OF THEORY AND EXPERIMENT

Fig. 2 presents top force versus time as measured for all experimental piles in the present study. Fig. 3 shows results for Pile 1, a full-scale steel pile 74 ft long. Theoretical and experimental results bear close resemblance, and the predicted static bearing capacity is \( C_0 = 225 \) kips as opposed to 214 kips measured in a static load test. \( C_0 \) is thus high by 5%.

\[ \text{REDUCED SCALE PILE NO. IX} \]
\[ C_0 + 9.9 \text{ KIPS} \]

\[ \text{VELOCITY IN FT/SEC} \]

\[ \text{TIME IN MILLISECONDS} \]

9.—ELASTIC AND RIGID BODY COMPONENTS OF VELOCITY AT TOP OF PILE

Fig. 4 shows results for Pile II, another full-scale steel pipe pile 74 ft long. Here the match of theoretical and experimental curves is less satisfactory but the static capacity prediction remains fair: \( C_0 = 290 \) kips as against a measured ultimate static value of 252 kips, the prediction error here being 15%.

Fig. 5 shows results for Pile III, with theory being a good match as to shape in displacement, velocity, and acceleration. The static capacity tilded is \( C_0 = 220 \) kips as opposed to a measured ultimate value of 204 kips, the dynamic result being high by 8%.

Fig. 6 presents results for Pile IIIA, the same pile as III but re-driven for an extended period of two weeks. The dynamic prediction of ultimate static capacity coincides with the actual static test result in this instance. This example typifies pile load capacity increase with time.

Because of the excellence of the dynamic prediction in this case it also becomes a good object for test of the sensitivity of the prediction method to the value of \( C_0 \) chosen. Fig. 7 illustrates this, where values of \( C_0 = 232, 242, \) and 252 kips have been tried, with corresponding theoretical displacements compared to measured test results. It is clear from this that close displacement matching sharply delineates the proper \( C_0 \) value under the present method.

The elastic and rigid body contributions to pile velocity may be separately calculated. The elastic result contains almost all of the "oscillatory" portion of the response, whereas the rigid body contribution is one of almost a straight-line decreasing velocity, suggesting the action of a rigid body under constant deceleration from constant soil resistance.

More can be said of the new information revealed by analyses of this type. This will be reserved to a later point, after a simplified dynamic pile calculation has been suggested.

Fig. 8 illustrates the dynamic prediction method for a case of a very short pile (12 ft). The match achieved is the worst of all cases treated as far as curve shape is concerned, though general trends are preserved. However, the predicted ultimate static bearing capacity is \( C_0 = 0.9 \) kips as opposed to a measured 9.5 kips, the prediction being high by 4%.

The calculated rigid body and elastic contributions for this case are presented in Fig. 9. Here, in this short pile, the elastic contribution is seen to be relatively much less important. As seen from this figure, the method under use clearly delineates the different roles of both elastic and rigid body motions in the action of piles.

SIMPLIFIED METHODS OF ANALYSIS

In Ref. 9 a rigid pile model was employed. Briefly, the analysis method used therein was as follows:

Let \( M \) be the mass of the pile and \( \ddot{X} \) its acceleration; then the force bal-

\[ F_T(t) = M \ddot{X} \]

\[ R_L = M \ddot{X} \]

\[ \text{FIG. 10.—TOTAL RESISTANCE} \]

\[ R^2(t) = F_T(t) = M \ddot{X} \]

\[ \text{OF A TYPICAL FULL-SCALE PILE} \]

\[ F_T(t) = R_L L = M \ddot{X} \]

\[ \text{MILLISECONDS} \]

\[ \text{TOTAL RESISTANCE IN KIPS} \]

\[ \text{2 4 6 8 10 12 14 16 18 20} \]

\[ \text{-400 -200 0 200 400} \]
which \( R_s \) is the soil resistance per unit length. Under the assumption that soil resistance has the simple form

\[
R_s \cdot I = C_o + a_1 \dot{X} + a_2 X^2 + \ldots
\]

\( \text{(4)} \)

be value of \( C_o \) may be determined experimentally if \( F(t) \) and \( \dot{X} \) are known at once instant when the velocity \( X = 0 \), at which instant \( R_s \cdot I = C_o \).

Use of this scheme does in fact yield reasonable order-of-magnitude results for \( C_o \) as indicated in Ref. 9. However, various causes act to reduce the accuracy of this method, the chief ones being the elastic action of the pile and the typically "oscillatory" and "shaky" appearance of the usual accelerometer record, which must be read to obtain \( X \) at the point where its time integral \( X \) passes through zero. The particular rapid local variations of \( X \) in the region of \( X = 0 \) render the ending of \( X \) there quite uncertain of interpretation.

One very useful scheme for the correction of these inaccuracies is to plot \( R_s \cdot I \) from Eq. 3 for all times \( t \) of interest to a given hammer blow (i.e., several tens of milliseconds). A typical result of this kind is shown in Fig. 10, beginning approximately at time \( t_1 = \frac{L}{c} \) and ending approximately

\[ \text{TABLE I.--BEARING CAPACITY RESULTS IN KIPS} \]

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>( C_o )</th>
<th>( R_s \cdot I_{eq} )</th>
<th>( R_s \cdot I)avg</th>
<th>Static Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>225</td>
<td>229</td>
<td>228</td>
<td>224</td>
</tr>
<tr>
<td>II</td>
<td>290</td>
<td>291</td>
<td>290</td>
<td>292</td>
</tr>
<tr>
<td>III</td>
<td>230</td>
<td>232</td>
<td>231</td>
<td>234</td>
</tr>
<tr>
<td>IIIa</td>
<td>242</td>
<td>245</td>
<td>244</td>
<td>242</td>
</tr>
<tr>
<td>IV</td>
<td>9.9</td>
<td>10.5</td>
<td>10.5</td>
<td>9.5</td>
</tr>
</tbody>
</table>

\( \text{Notes:} \) %n \( X \) passes through zero, or for any chosen shorter time interval, the average of \( R_s \cdot I_{eq}(t) \) is obtained. This average proves to be quite close to the measured or the dynamically calculated value of \( C_o \), especially if one takes into account the relative velocity of \( X \) is zero, and \( \dot{X} \) passes through zero. The time averaged can often be made quite well by eye by simply fitting a straight line to the data in the vicinity of \( t_0 \) and \( t_1 \).

This scheme is, in fact, little different in effect from another variant in which the velocity is used. A best-fit straight line is fitted to the descending portion of the velocity curve, as those in Figs. 3 through 6, and the slope of this is considered to be the rigid body acceleration; then using Eq. 3, the value of \( F_p(t) \) is read off at the time when the afterFit straight velocity line passes through zero, yields a value for \( R_s \).

That such approximate schemes may work quite well is attested to by the results portrayed in Table I. The first column identifies the piles discussed previously under the dynamic analysis method. Column 2 gives \( C_o \) as found by matching method. Column 3 gives the resistance calculated from only the rigid body portion of the predicted dynamic response at the point where the velocity passes through zero. Column 4 shows results of simplified averaging in the neighborhood of \( t_0 \) based on Eq. 3. Column 5 presents actual static test results for ultimate load bearing capacity.

It may be observed, however, that the very practical graphical aggregating methods suggested, do in fact, yield good results in the cases considered. However, further the more elaborate computer analysis unnecessary except as a research tool. This is a decided advantage, suggesting the possibility of rapid engineering estimation of pile static bearing capacity in the field through the use of instrumental dynamic tests.

**CONCLUSIONS AND DISCUSSION**

While earlier analytical methods have already been presented in the literature for analyzing pile driving "by the wave equation," the present method directly combines and compares theory with corresponding dynamic experiment. The method offers the possibility of dynamic in place of static load testing for piles, at least in certain types of soil.

In the present case, the measured hammer force is taken as input, while the field measured pile elastic and rigid body combined acceleration, plus velocity and displacement derived therefrom are considered as output. The analytical method postulates the same input and predicts the output. When the match between actual and predicted output is good, the prediction of static soil resistance also appears to be good. In this work a continuous pile model—rather than a set of discrete spring-masses—was used. While the continuous constant-section pile model has somewhat analogous advantages to those of a continuous constant-section beam model, this aspect of the present study is secondary to the main point. The use of any suitable analysis method for matching theoretical to experimental output will in principle be acceptable. What is considered novel in the present paper is the demonstration that, under suitable analysis, dynamic test information can yield the static bearing capacity. For example, it would appear that, in Refs. 2, 3, use could have been made directly of hammer force as portrayed in Fig. 2, instead of initial ram velocity as input. This might have the effect of increasing the accuracy of the calculated dynamic resistance as well as the ultimate static bearing capacity.

The present studies were made in the main on piles in silt, silt in soil (see Appendix IV). The favorable results obtained may be a direct result of this fact. An intentional effort was made, however, to predict results independently of detailed knowledge of the soil. One of the points of the study is that this may be possible for dynamic tests just as it is for static tests. To what extent this proves to be true for all types of soil is a subject for continuing study. It may well require more detailed regard to the definition of the assumed resistance law along the pile. It is hoped that light may be shed upon this by the experiences of others and that constructive comment may be elicited on this point. One of the additional questions is that of the pile toe resistance. For point bearing piles a form for this, \( F_p(t) \), must be assumed. The present study has succeeded to a reasonable extent with a very simple over-all resistance law. Continuing study may improve this law.

The attempt here to achieve simple dynamic estimates of bearing capacity through more easily applied criteria has also yielded encouraging results. In particular, the salient characteristics of the pile activity under the hammer
considered to be useful for analysis when there existed a net permanent pile settlement after impact. Full delineation of this minimum penetration amount is still open to further study.

The present paper lays no claim to solving the problem of determining static capacity by dynamic measurement under all conceivable circumstances. It does, however, present more than coincidental evidence that such a problem may be well-posed and that continued effort in this direction can result in even excellent and rapid dynamic predictions of pile static bearing capacity. The full range of applicability of the method as well as its limitations are objects of continuing study by the writers.

ACKNOWLEDGMENTS

A portion of the theoretical work and the entire experimental work reported here was sponsored at the former Case Institute of Technology by the Ohio Highway Department and the U.S. Bureau of Public Roads. The opinions, findings, and conclusions expressed in this paper are those of the writers and not necessarily of the State of Ohio or the Bureau of Public Roads.

The writers are particularly indebted to the reviewers of the original manuscript of the paper for their many constructive suggestions and comments.

APPENDIX I—THEORY OF THE ELASTIC PILE

The basic governing equation is the classical, one-dimensional elastic wave equation, written for the pile with a resistance term \( f_2/(AE) \) included.

\[
\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \frac{1}{3} \frac{\partial^2}{\partial x^2} \frac{\partial X}{\partial x} + \frac{R_L}{AE} \quad (0 < x < L) \quad (t > 0) \quad (5)
\]

In which \( c = \sqrt{E/\rho} \) is the wave velocity in the pile material, \( \rho \) is the pile material density, \( E \) is the modulus of elasticity of this material, and \( A \) is the pile cross-sectional area. (Pile dead weight is neglected in Eq. 5. It is a negligible fraction of the ultimate pile static capacity).

The boundary conditions are

\[
AE \frac{\partial^2}{\partial x^2} = - F_T(t) \quad (x = 0) \quad (t > 0) \quad (6a)
\]

\[
AE \frac{\partial^2}{\partial x^2} = - F_T(t) \quad (x = 0) \quad (t > 0) \quad (6b)
\]

The initial conditions are quiescent

\[
X(0) + \{x, 0\} = 0 \quad (0 \leq x \leq L) \quad (7a)
\]

\[
\dot{X}(0) + \{x, 0\} = 0 \quad (0 \leq x \leq L) \quad (7b)
\]

Through defining:

\[
Z(x, t) = X(t) + \{x, t\} \quad (8)
\]
The II Application of Soil Resistance Law

\[
\left\{ \begin{array}{l}
\omega \equiv J \equiv 1
\\
\gamma = \omega = r_0
\end{array} \right.
\]

\[
\left( 1 / 4 - \frac{1}{4} \right) \left. \frac{\partial}{\partial r} \right|_{r = 0} \left( \frac{T}{T'} \right) = (1^0 x)^{r_0}
\]

\[
(10)
\]

\[
(10') \left[ \frac{T}{T'} \right. \left( \frac{\partial}{\partial r} \right)_{r = 0} = (1^0 x)^{r_0}
\]

\[
(11)
\]

\[
(11') \left[ \frac{T}{T'} \right. \left( \frac{\partial}{\partial r} \right)_{r = 0} = (1^0 x)^{r_0}
\]

The II Application of Soil Resistance Law

In particular, at \( x = 0 \), the slope is 0.

Meaning Capacity

\[
\left( \frac{T}{T'} \right)_{x = 0} = \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^2) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^3) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^4) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^5) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^6) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^7) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^8) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^9) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

\[
(11''y^{10}) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

The governing equation becomes

\[
\left( \frac{\partial^2}{\partial x^2} \right) \left( \frac{T}{T'} \right)_{x = 0} = (1^0 x)^{r_0}
\]

March 1999
or \( K(x) \) a coefficient depending upon soil shear resistance at the pile-soil interface. For simplicity \( K \) was taken as a constant \( K = K_0 \) throughout the pile length. It may be pointed out in advance of the examples considered that this resistance law, essentially an "elasto-plastic" law, for time preceding \( t_c \), had small effect upon the response, whatever the value of \( K \) chosen, only causing small changes in the early and less significant portion of the calculated pile displacement response. The form of the corresponding soil resistance law as a function of time is suggested in Fig. 12. The values of \( Z \) used in the latter modified resistance law were simply taken as those measured at the top of the pile.

It should be remarked that certain series convergence problems connected with the computer evaluation of Eq. 14 necessitated special attention. These problems were essentially (a) by working directly with displacements in the form given by Eq. 17 from which the velocity and acceleration were obtained using a finite difference approximation scheme, and (b) by employing piecewise linear approximations to the given input functions \( H(t) \), together with an adequate number of terms to the series (Eq. 17).

**APPENDIX II.-INSTRUMENTATION**

To measure acceleration a quartz high-frequency accelerometer (Kistler 28A) was bolted directly to the side of the steel pipe pile, with a single bolt, just 2 or 3 pile diameters below the cap block position. Output wires from this led first to a charge amplifier, the signal from which then entered a power amplifier; this latter drove the light beam galvanometers of a Honeywell Visicorder which provided a photographic record on paper of events versus time. Galvanometers used were nominally of a 3000 cycle type with a resulting response rapid enough to follow the high frequency accelerometer goals. Typically, a high paper speed (80 in. per sec) was used when recording a pile hammer blow.

To measure hammer force a variety of procedures was tried, all centering around the use of foil resistance strain gages in a bridge-carrier amplifier configuration. At first strain gages were mounted directly on the outside pile surface. These proved satisfactory but as a rule delicate to apply under field conditions. Later a reliable force transducer bolted to the top of the pile, below the cap block and cushion assembly as shown in Fig. 13, was developed. This latter development allowed more nearly permanent strain gage instrumentation with the advantages of laboratory, as against field-installation, of the gages themselves. The entire force transducer was bolted to a steel plate which in turn was welded to the pile. The force transducer was statically calibrated periodically in the laboratory. Recording the hammer force during driving and upon restriking the pile was achieved by conditioning the strain signal with a carrier amplifier which in turn drove a galvanometer of the above-mentioned recorder.

The entire system was carried to the field in a small truck. Occasionally, portable a-c power was needed, but in general, conventional a-c power was available on all sites visited.

A point to be noted is that, while strain (force) and acceleration were desired for the top of the pile, it was necessary to make these measurements somewhat below the actual top (2 diam or 3 diam) to assure diffusing out of

**FIG. 12.-ALTERNATE APPLICATION OF SOIL RESISTANCE LAW**

**FIG. 13.-ASSEMBLY OF PILE DRIVING APPARATUS USED FOR RECORDING DYNAMIC IMPACT RESPONSES**
the hammer force uniformly into the pile body as well as to provide more safety for the accelerometer. It was also found useful to employ a double set of both strain gages and accelerometers and to average results. In the case of the accelerometers this provided some information on the effect of rocking of the top end of the pile. 

APPENDIX III.—PILE STATIC TESTS

The ML Test for static bearing capacity followed the Standard Ohio Highway Department load test procedure which is based on a variant of the "Engineering News Formula." In this procedure an initial load of 4/5 R is applied, in which the yield load is \( R = 2E_h/S + C_1 \); \( E_h \) = rated hammer energy (weight times drop height ft-lb); \( S \) = pile penetration in inches per blow; and \( C_1 \) = a coefficient lying between 0.10 and 0.30. Ultimate load is taken as six times \( R \). An additional increment of 1/5 \( R \) is applied after all measurable settlement has "ceased" (i.e., less than 0.01 in. in 20 min). Additional increments of 1/5 \( R \) are applied in similar fashion thereafter, the elapsed waiting time being increased one hr for each load increment. The ML tests were terminated, for this study, only after ultimate load had been reached.

Upon completion of the conventional ML Test, a CRP Test was performed. The penetration rate of the pile during this test is maintained at a predetermined rate, varying from 0.002 in. per min to 0.010 in. per min. Again, the load is maintained until ultimate capacity is reached.

Fig. 14 illustrates the typical form of load versus settlement plot obtained with the two types of test. Details may be found in Ref. 8.

### TABLE 2.—SOIL NEAR PILE I

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<th>Fine sand (3)</th>
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### TABLE 3.—SOIL NEAR PILE II

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The ultimate load as used in this study is defined as the value to which both ML and CRP Tests tended asymptotically in all cases.
APPENDIX IV.—SOIL CHARACTERISTICS

All of the piles reported on in this paper were driven in silty soil. This soil type is undoubtedly an important factor in the success of the prediction.

TABLE 4.—SOIL NEAR PILES III and IIIa

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<th>Depth, in feet</th>
<th>Aggregate</th>
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<td>28</td>
<td>40</td>
<td>19</td>
</tr>
</tbody>
</table>

schema employed to date. It remains to be shown that equal success can be gained with other soil types.

Details of the soil borings relative to each pile are given in Tables 2 through 5. In these Tables the following column headings and conventions are followed: Figures given for solids are percent by weight of total solids. Aggregate is defined as solids greater than 2 mm in size (i.e. not passing through 2 mm mesh openings). Coarse sand ranges from 0.42 to 2 mm; fine sand ranges from 0.074 to 0.42 mm; Silt lies between 0.005 and 0.074 mm; clay measures below 0.005 mm; Water content is given by percent weight of sample.

APPENDIX V.—REFERENCES

8. Tomkota, J., "Dynamic Studies on the Static Bearing Capacities of Piles," thesis presented to Case Western Reserve University, Cleveland, Ohio, September, 1963, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

APPENDIX VI.—NOTATION

The following symbols are used in this paper:

A = cross-sectional area of pile;
C = speed of sound in steel pile;
R = predicted ultimate static soil resistance;
D, d = damping coefficients;
E = elastic modulus of pile;
F = rated hammer energy;
F_T, F_B(t) = measured forces at top and bottom of pile;
Experimental study of buckling of buried domes

By Z. Getzler,¹ M. ASCE, and L. Lupu²

Introduction

Experimental and theoretical study of the combined action of soil and structure is a prerequisite in safe and economical design of a buried structure. When calculating the loads involved, it should be borne in mind that the two components form a single integrated system, with the soil serving as the medium transmitting the external forces to the structure and protecting it by arching and restraint.

The spherical dome is an example of the ideal structure of this type, capable of supporting high loads through axial-membrane stresses alone; however, the danger of buckling is an adverse factor preventing full utilization of the cross section, especially in cases of thin elements or ones with a low modulus of elasticity.

Herein the writer describes buckling tests carried out on model domes buried in sand; results are compared with recent data obtained for unburied ("free") domes under hydrostatic load.

Buckling of unburied domes

The problem of buckling of spherical domes under uniform radial load, with small, linear, symmetric deformation, was solved by Zoelly (1915) and Schwerin (1922). The classical derivation (9)² yields

\[ P_{th} = \frac{2}{\sqrt{3(1 - \nu^2)}} E\left(\frac{t}{R}\right)^2 \]

(1)

Note.—Discussion open until August 1, 1969. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Soil Mechanics and Foundations Division, Proceedings of the American Society of Civil Engineers, Vol. 70, No. SM2, March, 1969. Manuscript was submitted for review for possible publication on April 25, 1968.

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³Numerals in parentheses refer to corresponding items in the Appendix I.—
References.
PILE FOUNDATION ANALYSIS

William E. Saul
PILE FOUNDATION ANALYSIS

William E. Saul, 1 Fellow ASCE

The design of pile foundations for static or dynamic loads can be accomplished through the stiffness method of analysis utilizing soil-pile interaction models derived from the beam-spring foundation idealization although any linear or piecewise linear model will suffice. The model advocated may be of finite length or made up of a series of finite lengths due to variation in the soil or the pile.

INTRODUCTION

The structural analysis of pile foundations requires computation of displacement of the pile cap and forces on and displacements of each of the piles. A lengthy set of references to this problem were given by Saul, 1968(1), in a paper which presented the basic framework of the material contained herein. Subsequent additions by O'Neill(2), Murthy and Shrivastava(3), Prakash and Chandrasakaran(4), Saul(5,6) and Saul and Wolf(7,8) point out shortcomings to the original paper and supplement it considerably. Formulation of the problem and the computational techniques have not been found lacking; see Vesic(9), Bowles(10), and Arya, O'Neill and Pincus(11). Improvements and discussion of the pile-soil interaction modeling, however, continue to appear; see references (9,11) for general discussions and summaries. Major work is being done by Vesic(9), Poulos(12) and by Novak(13) and their colleagues, where the references cited are samples of their extensive works, on pile-soil interaction. The primary thrust in these works, however, has been derived from elastic theory although in a more complete and complex sense than the spring foundation idealization. This leads to a better understanding of the problem; but as yet, with soil properties normally available, it is questionable whether any model is better suited for design and analysis in practice. Interaction of the pile cap with the soil has been investigated by O'Rourke and Dobry(14). When this factor is to be included the stiffness contribution of the soil-cap may be directly added to the foundation stiffness matrix as computed herein.

The objectives in this paper are to: 1. Update the material presented earlier(1). 2. Include improvements developed since(5,6,8).

1 Professor, Department of Civil and Environmental Engineering, The University of Wisconsin, Madison 53706.
3. Present a major development which allows inclusion of layered soils, piles which may vary in section properties with length, or short piling (9). And, 4. Give examples to illustrate the computational method.

A DEMONSTRATION

Consider a pile foundation consisting of a rigid reinforced concrete cap and any number of attached piling. The piling may be vertical or battered, short or long, at the same reference elevation within the cap or at several levels and of similar or different materials, sections or sizes. The piles may have been placed in any manner or by different methods at the same time or at various times. Their principal axes may be at any angle and have any degree of fixity between hinged and rigid with the cap. The cap is assumed rigid (in later computations), but may be of any shape and thickness, including variable thickness such as stepped, and may be constructed so that the piles are embedded at different elevations within the cap. The soil may be homogeneous, varying or layered, including layers of very weak or negligible soils. The pile cap may be in contact or embedded in the top layer of soil or elevated as a platform. In summary, for the experiment, there are nearly no constraints.

To proceed, a coordinate center for the pile foundation is assigned and a set of Cartesian coordinates. Although these may be assigned arbitrarily, it is useful to choose an origin at least vertically aligned with the center of mass of the foundation and/or load and have one horizontal axis parallel to an axis of symmetry of foundation or loads, should one exist.

The demonstration is to load the foundation in each of six components, one component at a time. These six are the three rectilinear forces and the three moments corresponding to the axis just established as shown in Fig. 1. When one of these forces $Q_i$ is applied, and none other, the six corresponding components of deflection, $D_i$, 3 rectilinear and 3 rotational, may be measured and plotted. These forces may be applied by increments of force, i.e., dead loads, or displacement, i.e., by jacking, by smaller or larger increments, slowly or more rapidly, and monotonically or cycled. The resulting curves may be nonlinear with the rate of deflection increasing with higher loads. Creep or relaxation, depending on the type of load test, may show a time dependency, i.e., a viscoelastic material. In many cases, the load-deflection curve will exhibit a regime with a slowly changing or almost constant slope followed by a regime with a rapidly changing slope,
Fig. 1 - Foundation Loads and Displacements

Fig. 2 - Pile Forces and Displacements
and then another regime with a very steep, perhaps again constant, slope. Although this may appear as a typical stress-strain curve with an elastic, yield and plastic behavior such is most probably not the case and it may often be difficult to distinguish the three zones. The reason is that soil is often viscoelastic and that there are size effects, edge effects, possible friction between cap and soil, soil pressures developing against embedded caps, and any number of other possible influences which could inhibit reproducibility of test results. Nevertheless, it is suggested that if the load were cycled several times in the neighborhood of magnitude which is eventually expected, that a nearly constant slope to each curve would be found. If the fluctuation of load magnitude was expected to be wide, the curve could be approximated as bilinear or piecewise linear. Thus, each of the 6 loading conditions produces 6 deflections $d_{ij}$ where $d_{ij} = D_{ij}/Q_j$; that is, the deflection in direction $i$ due to a unit load in direction $j$, $d_{ij}$, is the measured deflection at $i$, $D_{ij}$ divided by the load $Q_j$ at $j$. The 6 by 6 matrix of these flexibility influence coefficients $[d]$, where each column $[d]_j$ is produced by one load, is the structural flexibility matrix. It may be inverted to obtain the structural stiffness matrix $[S]$, $[S] = [d]^{-1}$, where each coefficient $S_{ij}$ is the force at position $i$ due to a unit displacement at position $j$ with all other displacements equal to zero.

It is to be observed that the slope determined upon immediate loading of a pile foundation, or with a light loading, or by using a different foundation design, such as a scaled-down configuration, would be different, usually stiffer, than the value obtained as described earlier. Such a test would be expensive, time consuming, and true only for that foundation in that place. However, the significance is in understanding the nature of the structural stiffness matrix since it is a real and important property of the foundation, in fact, the foundation's signature. It relates forces, or loads, applied to the foundation and the resulting displacements in the coordinates already established. Thus,

$$Q = [S][\Delta]$$

(1)

where $(Q)$ are loads and $(\Delta)$ deflections of the foundation. The components of $[S]$ are constant if the system is linear or quasi-linear as described earlier; otherwise they are variables which may be taken as piecewise linear.

**PILE BEHAVIOR**

Since determination of the structural stiffness matrix $[S]$ by experiment has the limitations noted a design alternative is necessary and
provided. It is first useful to consider a similar demonstration conducted on a single pile. The pile may be placed in any manner, be of any shape or materials, any length, be placed in any type of soil or soils, and be flush with the surface or extend into the air. For this demonstration the pile should be vertical. Coordinate axes are chosen along the longitudinal centroidal axis and the principal axes of bending. Applying 6 loads, along each axis and a moment about each axis, one at a time, as shown in Fig. 2 results in a displacement vector \( \{G\}_i \) for each load. Although vector \( \{G\}_i \) has 6 components, most will be zero since flexure about a principal axis should eliminate out-of-plane, torsional and axial components. Thus, the axial and torsional loads are expected to result in only axial and torsional displacements, respectively, and the flexure-producing loads in only 2 component displacements each. The matrix \( [G] \) then is quite sparse, with only slight flexural coupling. Once again, the soil-pile interaction is nonlinear so it would be useful to cycle the load in the neighborhood of magnitude of the loads expected so that a realistic linear approximation can be achieved between load and deflection. Once this constant slope is selected, dividing the measured displacements \( \{G\}_i \) by the magnitude of the load in direction \( i \), the resulting displacements are the pile flexural influence coefficients \( g_{ij} \), the deflection in the direction \( i \) to a unit force in direction \( j \) with all other forces zero. The pile flexibility matrix \( [g] \) may be inverted to obtain the pile stiffness matrix \( [b'] \) where \( b'_{ij} \) is the force in direction \( i \) due to a unit displacement in direction \( j \) with all other displacements zero. The form of these pile matrices would be

\[
[g] = 
\begin{bmatrix}
  g_{11} & g_{15} \\
  g_{21} & g_{24} \\
  g_{31} & g_{33} \\
  g_{41} & g_{44} \\
  g_{51} & g_{55} \\
  g_{61} & g_{66}
\end{bmatrix}
\quad
[b'] = 
\begin{bmatrix}
  b'_{11} & b'_{15} \\
  b'_{21} & b'_{24} \\
  b'_{31} & b'_{33} \\
  b'_{41} & b'_{44} \\
  b'_{51} & b'_{55} \\
  b'_{61} & b'_{66}
\end{bmatrix}
\]  

(2)

and the forces \( \{F\}_i \) and displacements \( \{x\}_i \) acting on pile \( i \) are related by

\[
\{F\}_i = \{b'\}_i \{x\}_i \quad \text{and} \quad \{x\}_i = [s]_i \{F\}_i
\]  

(3)

**GROUP ACTION**

When placed into the foundation coordinate system, as shown in Fig. 3, the pile may be battered, that is, placed at an angle \( \gamma \) with
Fig. 3 - Pile Cap with Pile i
the vertical or on a batter slope of $1/h_1$ where cotangent $Y_1 = h_1$ and 
$c_1$ is the clockwise angle to the direction of batter from the $U_1$ axis of 
the foundation in plan view. Further, the pile head is located at 
coordinates $U_1(u_1, u_2, u_3)$ with respect to the cap coordinate system and 
the pile's principal axes may be rotated to an angle $c_1$ with respect 
to a coordinate system described by the vertical plane containing the
battered pile where $u_1'$ is perpendicular to this plane and in the hori-
izontal $U_1$ plane and $u_2'$ is perpendicular to $u_1'$ and the longitudinal axis
$u_3'$ of the pile. Using the appropriate transformations, the stiffness 
matrix of the pile with respect to the foundation coordinate system $U_1$ is

$$[S']_1 = [c]_1 [a]_1 [b']_1 [p]_1^T [s]_1 [c]_1^T$$  \hspace{1cm} (4)$$

where

$$[p]_1 = \begin{bmatrix} [p'] & [0] \\ [0] & [p'] \end{bmatrix}$$  \hspace{1cm} (5)$$

$$[p']_1 = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (6)$$

$$[c]_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -u_3 & u_2 & 1 & 0 & 0 \\ u_3 & 0 & -u_1 & 0 & 1 & 0 \\ -u_2 & u_1 & 0 & 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (7)$$

$$[a]_1 = \begin{bmatrix} [a'] & [0] \\ [0] & [a'] \end{bmatrix}$$  \hspace{1cm} (8)$$

and

$$[a'] = \begin{bmatrix} \cos \gamma \cos \phi & -\sin \gamma \sin \phi & \sin \gamma \\ \cos \gamma \sin \phi & \cos \gamma \cos \phi & -\cos \gamma \\ -\sin \gamma & 0 & \cos \gamma \end{bmatrix}$$  \hspace{1cm} (9)$$

which may be written

$$[S']_1 = [cap]_1 [b']_1 [cap]_1^T$$  \hspace{1cm} (10)$$

The stiffness matrix of the foundation is the sum of the stiffness of all
the piles $n$ in the foundation.
\[
[S] = \sum_{i=1}^{n} [S_i]^T
\]  

(11)

Thus, from Eq. 1 the loads may be determined for a given displacement \( \Delta \) of the foundation or the displacement determined for a given load \( q \).

Once the foundation displacements \( \Delta \) are determined, the forces and displacements of individual piling may be calculated in the coordinate system parallel to the foundation coordinate \( U \) from

\[
\{x\}_4 = [c]^T_4 [\Delta] \quad \text{and} \quad \{F\}_4 = [ap]^T_4 [b']_4 [ap]^T_4 [c]_4^T [\Delta] 
\]  

(12)
or in member principal axes from

\[
\{x\}_4 = [cap]^T_4 [\Delta] \quad \text{and} \quad \{F\}_4 = [b']_4 \{x\} = [b']_4 [cap]^T_4 [\Delta] 
\]  

(13)

when either member principal axis is horizontal \([p] = [1]\) and, therefore, the rotated member stiffness matrix \([b]\), where

\[
[b] = [p][b'][p]^T
\]  

(14)

becomes identical with \([b']\). Elements of Eq. 10 with \([b'] = [b]\) are presented in analytic form in Appendix I.

SOIL-PILE INTERACTION MODELS

The components of the pile stiffness matrix \([b']\), Eqs. 2 and 3, may be obtained by experiment, as noted, but analytic models using readily available soil data are necessary. Herein five analytic models are presented, all based on the spring foundation idealization for lateral loading (flexure).

It may be assumed that in the neighborhood of interest, i.e., load magnitude and pile dimensions, the soil's reactive pressure on the pile is linearly proportional to the deflection, thus, defining a value \( k_s \) in units of pressure per unit deflection such as \( \text{lb/ft}^3 \) or \( \text{N/m}^3 \), which is a property of the soil. The pressure then is \( k_s x \), which is analogous to a linear spring. Considering the pile as a beam the reactive force per unit length of beam may be expressed as \( k_s D_1 z_1 \) where \( D_1 \) is the projected width of the beam in the direction of bending. Now \( k_s \) is not a primary soil property but may be expressed in terms of elastic constants (15) or adjusted for size and shape (16) from test data. Thus, it may be that the product \( k_s D_1 z_1 \) is more meaningful than \( k_s \) alone. Since the spring concept neglects shear coupling in the idealization, it appears that there certainly should be considered an angle effect as well as a direct spring effect. This, however, is done
when tests with a unit plate size are used as a standard to determine $k_s$ and this value is then adjusted for shape and size.

The subgrade modulus also varies with depth because of confinement of the soil, soil properties, and possible variation of soil with depth, i.e., layering (17). It is easily postulated and confirmed by measurement that the soil at surface about a free-standing pile has no vertical constraint and, therefore, cannot support even low values of horizontal pressure. However, presence of a cap may provide the constraint or account for the cap-soil friction. The subgrade modulus for an over-consolidated cohesive soil appears to approach a constant value with depth once out of range of surface effects, i.e., $k_s^D$, a constant. However, in granular or normally loaded cohesive soils, the subgrade modulus increases with depth and may be assumed to do so linearly so that $k_s = \eta z$ where $z$ is depth of soil and the soil parameter $\eta$ has the units of lb/in.$^4$ or N/mm.$^4$.

Solution of the beam equation introduces parameters $\beta$ and $\psi$, where

$$\beta = \frac{k_s^D}{4E} \quad \text{and} \quad \psi = \frac{\eta d}{E}$$

which have units of per unit length, i.e., in.$^{-1}$ or mm.$^{-1}$. Note that these parameters are directional, i.e., may be different with respect to each principal axis if $I_1$ and or $D_1$ are not equal. If $\beta L$ or $\psi L > \pi$, where $L$ is the embedded length of the pile, the pile may be considered as being long or infinite in length; which means that lateral deflection has been effectively damped to negligible above the pile tip. For the case of a long pile four models for the flexural stiffness coefficients are presented in Appendix II. They are:

A1. A single layer of soil with $k_s$ a constant.

A2. A single layer of soil with $k_s = \eta z$, i.e., increasing linearly with depth.

B1. A two-layer system where $k_s = 0$ in the top layer because of an elevated cap (platform), negligible or poor soil, or lack of confinement to develop an effective lateral soil pressure. The lower layer has $k_s$ a constant.

B2. A two-layer system where $k_s = 0$ in the top layer, similar to B1 and $k_s = \eta z$ in the lower layer; $z = 0$ at the top of the second layer.

When $\beta L$ or $\psi L$ lateral displacement of the pile tip may occur and the stiffness coefficients which apply for long piles become less useful. The smaller $\beta L$ or $\psi L$ the more pronounced the effect. In addition to short or intermediate length piles, nonlinear variation of
with depth, layering or strata of the soil, or variations in the pile section properties or materials along its length provide application for a fifth model based on a layered system. Model C is based on a beam or spring foundation element of finite length with 8 lateral kinematic degrees of freedom (2 rectilinear displacements and 2 rotational displacements at each end), see Appendix III. It is assumed for model C that each section or element $k_{e1}$ is constant and the pile is prismatic; however, each element may have different soil and pile properties, including $k_e = 0$, and there may be anywhere from one to a large number of segments. A pile of $n$-segments will have $4(n+1)$ lateral degrees of freedom in displacement. Since the axial degrees of freedom, compression and torsion, are not coupled with the flexural they are included later although they could easily be included at this stage. In the computation the 8 by 8 stiffness matrix shown in Appendix III is computed for each segment and the element stiffness matrices summed to produce a $4(n+1)$ square stiffness matrix of the pile. This matrix is then condensed to eliminate all degrees of freedom except those of interest [b'], at the pile head. The large matrix may be retained if later computation for stress resultants, i.e., displacements, shear, moments, are desired along the pile. The procedure outlined for model C is best accomplished through use of a computer.

The factor $\delta$ used in expressions for the pile stiffness coefficients $b'_{ij}$ in flexure is a measure of the connectivity of the pile to the pile cap. Thus, $0 \leq \delta \leq 1.0$ where $\delta = 0$ for a pinned or hinged condition and $\delta = 1.0$ for a fixed condition. If the connection is semi-rigid, i.e., $0 < \delta < 1.0$, the value of $\delta$ may be estimated. With model C, the layered or finite length pile, $\delta$ should be formulated as a multiplier to the coefficients computed earlier. These multipliers are $\delta$ for $b'_{44}, b'_{56}, b'_{15}$, and $b'_{24}$; for $b'_{11}$ and $b'_{22}$ it is 0.5 (146).

The longitudinal member stiffness coefficients, $b'_{33}$-axial and $b'_{66}$-torsion have been suggested (1) in the form

$$b'_{33} = \frac{k_L AE}{L} \quad \text{and} \quad b'_{66} = \frac{\delta k_T JG}{T \cdot L}$$

where $AE/L$ and $JG/L$ are axial and torsional member stiffness respectively, $\delta$ has been defined earlier, and the coefficients $k_L$ and $k_T$ are empirical participation factors. If the pile were fixed at its end only, these factors would be 1.0; however, they may differ considerably. O'Neill (2) notes that $k_T$ may be much larger, larger than the 2.0 previously suggested. The factor $k_L$ may be less or greater than 1.0. If the pile tip

10
did not move and relied on friction for bearing $k_L$ would be about 2.0.

Movement of the pile tip, however, acts to decrease $k_L$. Works which
may be consulted include those by Novak(18), Poulos(19), and Randolph
and Wroth(20) for discussion or alternative calculation of $b'_{33}$. For
a moving tip or layered system best evaluation of the longitudinal
stiffness appears to be essential. In a layered system the relation-
ship between load on the pile and displacement at the top of the
layer, where displacement includes rigid body movement of the pile
because of soil shear and shortening of the pile due to elastic com-
pression, may be denoted by $b'_{33i}$ for layer i. The stiffness coeffi-
cient of the pile in n layers is then calculated from

$$\frac{1}{b'_{33}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b'_{33i}} = \frac{1}{b'_{33,1}} + \frac{1}{b'_{33,2}} \ldots$$

(17)

A similar calculation can be made to determine the torsional stiffness
coefficient, $b'_{66}$, in a layered system.

**COMPUTATIONS**

A 10 inch XH pipe pile was used in computations to compare pile
stiffness properties using the finite length pile, Model A, and the long
pile, Model A1. Results are given below:

<table>
<thead>
<tr>
<th>Length (ft.)</th>
<th>$b'_{11}$ (k/in.)</th>
<th>$b'_{44}$ (in-k/radian)</th>
<th>$b'_{15}$ (k/rad. or in.-k/in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>51.31</td>
<td>9,821</td>
<td>614.2</td>
</tr>
<tr>
<td>4</td>
<td>95.10</td>
<td>69,494</td>
<td>2196.3</td>
</tr>
<tr>
<td>6</td>
<td>114.02</td>
<td>157,355</td>
<td>3461.2</td>
</tr>
<tr>
<td>10</td>
<td>117.04</td>
<td>212,505</td>
<td>3598.2</td>
</tr>
<tr>
<td>18</td>
<td>125.83</td>
<td>215,886</td>
<td>3683.3</td>
</tr>
<tr>
<td>24</td>
<td>128.06</td>
<td>216,786</td>
<td>3696.9</td>
</tr>
<tr>
<td>infinite</td>
<td>126.08</td>
<td>216,802</td>
<td>3697.0</td>
</tr>
</tbody>
</table>

Pile properties are: $A = 16.1 \text{ in.}^2$, $I_x = I_y = 211.9 \text{ in.}^4$, $D_x = D_y =
10.75 \text{ in.}$, $E = 30,000 \text{ ksi}$, and $G = 12,000 \text{ ksi}$. Soil modulus $k_s = 0.2$
kci. It can be seen that there is a rapid change after 6 ft. with the-
stiffness coefficients approaching the value for infinite. Note that
$\beta = 0.0171/\text{in.}$, thus $\beta L = \pi$ yields $L \geq 15.35$ ft. as a long pile.

If the above pile is topped with a 2 ft. cantilever, i.e., a 26 ft.
pile with 24 ft. embedded and 2 ft. in air, the coefficients are:

$\begin{align*}
&b'_{11} = 78.82 \text{ k/in.}, \quad b'_{44} = 211,508 \text{ in.-k/rad.}, \quad \text{and}\quad b'_{15} = 3257.1 \text{ k/rad.} \\
&\text{Obviously, the condition of the top layer is of major importance.}
\end{align*}$

A problem was solved, see Fig. 4, to illustrate the method of
computation. The pile properties are:
Fig. 4 - Example Problem

12
The soil subgrade modulus $k_s = 0.1$ kci. A foundation load of

$$ [q]^T = [40, 20, 600, 0, -500 \text{ in}-k, 0] \text{ was used.} $$

Results are as follows: (all units in kips and inches)

1. Pile stiffness values,

<table>
<thead>
<tr>
<th>Pile</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$b_5$</th>
<th>$b_6$</th>
<th>$b_7$</th>
<th>$b_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>74.90</td>
<td>74.96</td>
<td>120.75</td>
<td>182.302</td>
<td>182.302</td>
<td>12.714</td>
<td>2614.0</td>
<td>-2614.0</td>
</tr>
<tr>
<td>2</td>
<td>57.00</td>
<td>57.00</td>
<td>424.1</td>
<td>64.291</td>
<td>64.291</td>
<td>4.071</td>
<td>1353.6</td>
<td>-1353.6</td>
</tr>
<tr>
<td>3</td>
<td>58.99</td>
<td>78.40</td>
<td>1257.0</td>
<td>230,558</td>
<td>102,426</td>
<td>11,859</td>
<td>1736.1</td>
<td>-3063.3</td>
</tr>
</tbody>
</table>

2. The foundation stiffness matrix,

$$ [S] = \begin{bmatrix} 308.06 & 26.93 & 276.97 & \text{SYM} \\ 435.18 & 74.24 & 2704.86 \\ 2497.66 & 2600.15 & -1699.42 & 1,277,722 \\ 327.69 & -9410.49 & -9449.02 & -1,202,452 & 1,905,765 \\ 756.74 & 421.86 & 6912.82 & -285,315 & 189,448 & 327,523 \end{bmatrix} $$

3. The foundation displacements,

$$ [\Delta]^T = [-0.5132, 0.4991, 0.3741, 0.009045, 0.010398, -0.005489] $$

4. The pile displacements in member coordinates,

$$ [x_1]^T = [-0.2441, 0.4742, 0.1749, 0.014099, 0.004483, -0.001086] $$

$$ [x_2]^T = [-0.5931, 0.5686, 0.3741, 0.009045, 0.010398, -0.005489] $$

$$ [x_3]^T = [-0.8944, -0.2291, 0.1899, -0.003017, 0.013032, -0.006412] $$

5. The pile forces in member coordinates,

$$ [F_1]^T = [-6.58, -1.31, 211.2, 1330.8, 179.2, -13.8] $$

$$ [F_2]^T = [-19.72, 20.16, 158.7, -188.1, -134.3, -22.3] $$
\( \{F\}_3 = [-30.11, -8.89, 238.8, -6.8, -219.7, -76.0] \)

Although a computer program 2 was used to solve the problem, the stiffness matrix can be assembled using the equations in Appendix I. The member stiffness matrices \([k']\) can be determined from model A1 but were computed for this problem using C to check A1.

CONCLUSIONS

The spring foundation model has a number of advantages for modeling lateral loading including the ability to compute deflections, shears, bending moments and stresses along the pile. The computations are straightforward and understandable. Work remains to be done on improving the soil-pile interaction models and adding to the library of models available to the designer.

It is useful to realize that when piling are hinged or vertical, a number of variables become zero. In addition, the \([k']\) matrix may be the same if several or all piles in the foundation are the same. Further, symmetry of the arrangement of piling in a foundation may allow a decrease in the amount of computations. Symmetry of loading may also shorten computations, especially if a combination of geometry and loading allow the foundation to be analyzed as a plane figure.

Partial constraint of the pile to the cap is accounted for through choice of \(\delta\) between 0 and 1.

The addition of the finite length model, C, herein, is an important and valuable step forward allowing use of short piles, variable or layered soils, and variable section piles.

---

2 Program FILEFDN, UW Department of Civil and Environmental Engineering.
APPENDIX I - FORMULAS FOR STIFFNESS INFLUENCE COEFFICIENTS

Formulas are given for single piles. Stiffness coefficient $S'_{ij} = S'_{ji}$ by reciprocity and function $B_i$ are defined for convenience as follows:

$$B_1 = b_{11} \cos^2 \gamma - b_{22} + b_{33} \sin^2 \gamma$$
$$B_2 = (b_{11} - b_{33}) \sin \gamma \cos \gamma$$
$$B_3 = (b_{13} + b_{24}) \cos \gamma \sin \alpha$$
$$B_4 = u_1 \sin \alpha - u_2 \cos \alpha$$
$$B_5 = b_{11} \sin^2 \gamma + b_{33} \cos^2 \gamma$$
$$B_6 = b_{44} \cos^2 \gamma - b_{55} + b_{66} \sin^2 \gamma$$
$$B_7 = u_3 (b_{22} + b_{44}) \cos \alpha$$
$$B_8 = u_3 (b_{22} + b_{55}) \sin \alpha$$
$$B_9 = u_1 u_3 \sin \alpha \cos \alpha$$

Thus $S'_{11} = B_{10} \cos \alpha + b_{22}$

$S'_{12} = B_{12} B_{18}$

$S'_{13} = -B_{11}$

$S'_{14} = -u_2 B_{11} - B_9 B_3$

$S'_{15} = u_1 B_{11} + B_7 + B_{14}$

$S'_{16} = B_4 B_{10} - u_2 b_{22} + b_{24} \sin \gamma \sin \alpha$

$S'_{22} = B_{12} \sin \alpha + b_{22}$

$S'_{23} = -B_{13}$

$S'_{24} = -u_2 B_{13} - B_8 - B_{16}$

$S'_{25} = B_3 + B_9 + u_1 B_{13}$

$S'_{26} = B_4 B_{12} + B_{17}$

$S'_{33} = B_5$

$S'_{34} = u_2 B_5 + u_3 B_{13} + B_{15} \sin \alpha$

$S'_{35} = -u_1 B_5 - u_3 B_{11} - B_{15} \cos \alpha$

$S'_{36} = -B_2 B_4$
\[ S'_{44} = u_1^2 B_3 + 2u_2 B_{15} \sin \alpha + B_5 \cos^2 \alpha + B_{55} u_3(2u_2 B_{13} + B_3 + 2B_{16}) \]
\[ S'_{45} = -u_1 u_2 B_5 + B_{18} - B_{15}(u_1 \sin \alpha + u_2 \cos \alpha) - u_3(u_2 B_{11} + u_1 B_{13} + u_3 B_{18} + 2B_{33}) \]
\[ S'_{46} = u_2 B_3 - B_{24} - B_{19} \cos \alpha - u_3(B_{12} B_4 + B_{17}) \]
\[ S'_{55} = u_1^2 B_5 + B_5 \sin^2 \alpha + B_{35} + 2u_2 B_{15} \cos \alpha + u_3(B_7 + 2u_2 B_{11} + 2B_{18}) \]
\[ S'_{56} = u_1(B_2 B_3) - B_{19} \sin \alpha - u_3 B_{14} + u_3 [\sin(\gamma B_1 \cos \beta + B_{24} \sin \alpha)] - u_2(B_{10} \cos \beta B_{33}) \]
\[ S'_{66} = B_2 \beta_4 + (u_1 u_2 B_{22} - 2B_{24}) (u_1 \cos \alpha + u_2 \sin \alpha) \sin \gamma + (b_{44} - b_{66}) \sin^2 \gamma + b_{66} \]

where \( U_1(u_1, u_2, u_3) \) are the coordinates of the pile top in the foundation, \( \alpha \) is the angle to the direction of batter clockwise in plan from the \( U_1 \) axis and \( \gamma \) is the angle of batter from the vertical in the plane of batter.
Model

A. Single Layer

1. Beam on a Constant Spring Foundation
   \[ b''_{11} = \frac{(1+\delta) \kappa \beta}{k_1} \]
   \[ b''_{12} = \frac{\delta \kappa}{k_1} \]
   where \( \kappa = 28 \frac{EI}{I} \)

2. Beam on a Linearly Increasing Spring Foundation
   \[ b''_{44} = \frac{(0.427+0.6726) \lambda^2}{k_1} \]
   \[ b''_{55} = \frac{1.5036 \lambda}{k_1} \]
   where \( \lambda = \frac{EI}{I} \)

\[ b''_{15} = b''_{15} = \frac{\delta \lambda}{k_1} \]

B. Two Layer (if is the unsupported length of cantilever)

1. Cantilever Beam Adjoining Model A1
   \[ b'' = \frac{3[1+\delta(1+2 \beta \lambda)] k^2 \kappa}{k_1} \]
   \[ \delta(3+68 \lambda+66 \lambda^2 2) \]
   \[ 3(1+2 \delta \lambda+\beta \lambda^2 2) \]
   \[ 3(1+2 \delta \lambda+\beta \lambda^2 2) \beta \kappa_1 k_1 \]

   where \( k_1 = 1/(3+68 \lambda+66 \lambda^2 2+(1+\delta)28 \lambda^2 3+66 \lambda^2 4) \)

2. Cantilever Beam Adjoining Model A2
   \[ b''_{11} = \frac{3(1+2 \delta) [\psi_1 + \delta(1.686+\psi_2)]}{k_1} \]
   \[ 6(6.918+9.204 \psi_1) \]
   \[ 6(3.068+3.732 \psi_1 \lambda+r_1^2 \lambda) \lambda \psi_1 \]
   \[ +5.058 \psi_1^2 \lambda^2+\psi_1 \lambda^3) \lambda \psi_1 \]

   where \( s = 1/(18.176+1+\delta \lambda) \) \[ 6.918(1+2 \delta) \psi_1 +9.204(1+\delta) \psi_1 \lambda+1.686(3+\delta) \psi_1 \lambda^2+\psi_1 \lambda^3) \]

Subscripts \( i \) must be adjusted for the direction or axis of bending. For the fixed condition \( \delta=1 \) and for the pinned end \( \delta=0 \).

** The form of \( b''_{14} \) is always the same as the form for \( b''_{15} \) but negative. The values may not be the same however since \( \psi_1 \neq \psi_2 \) or \( \beta_1 \neq \beta_2 \) unless piles have \( I_x=I_y \) and \( D_x=D_y \).
APPENDIX III - STIFFNESS MATRIX FOR A PILE SEGMENT

\[
[k] = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & T_{3y} & 0 & 0 & T_{5y} & T_{4y} & 0 & 0 & -T_{6y} \\
2 & 0 & T_{3x} & -T_{5x} & 0 & 0 & T_{4x} & T_{6x} & 0 \\
3 & 0 & -T_{5x} & T_{1x} & 0 & 0 & -T_{6x} & T_{2x} & 0 \\
4 & T_{5y} & 0 & 0 & T_{1y} & T_{6y} & 0 & 0 & -T_{2y} \\
5 & T_{4y} & 0 & 0 & T_{6y} & T_{3y} & 0 & 0 & -T_{5y} \\
6 & 0 & T_{4x} & -T_{6x} & 0 & 0 & T_{3x} & T_{5x} & 0 \\
7 & 0 & T_{6x} & T_{2x} & 0 & 0 & T_{5x} & T_{1x} & 0 \\
8 & -T_{6y} & 0 & 0 & -T_{2y} & -T_{5y} & 0 & 0 & T_{1y} \\
\end{bmatrix}
\]

where, 
\( T_{1} = (C'S' - CS)\kappa q \)
\( T_{2} = (C'S - CS')\kappa q \)
\( T_{3} = 2(CS + C'S')\xi^2 q \)
\( T_{4} = 2(C'S + CS')\xi^2 q \)
\( T_{5} = (S^2 + S')^2 \kappa q \)
\( T_{6} = 2SS' \xi q \)

\( q = \frac{1}{(S^2 - S')} \)
\( C = \cos BL \)
\( S = \sin BL \)
\( C' = \cosh BL \)
\( S' = \sinh BL \)
\( \kappa = 2\beta EI \)
\( \beta'^4 = k_a D/(4EI) \)

Segment i of Pile with Member Degrees of Freedom

18
APPENDIX IV - REFERENCES


Errata & Discussions Relating to Paper
"Static and Dynamic Analysis of Pile Foundations"
by William E. Saul
Published in the Structural Journal, ASCE, May 1968, pp. 1077-1100

1. Errata:

pp. 1080-81-82 1. Change subscripts in expression for \( b_{11} \) to 2 in Eqs. 3, 13 & 19.
2. Change subscripts in expression for \( b_{22} \) to 1 in Eqs. 4, 14, & 20.
3. Change subscripts in expression for \( b_{15} = b_{51} \) to 2 in Eqs. 9, 17, & 23.
4. Change subscripts in expression for \( b_{24} = b_{42} \) to 1 in Eqs. 10, 18, & 24.
5. Change expression in Eqs. 19 & 20 to \((1 + 2 \delta_i)\Delta\) (i.e., change - to +.)

p. 1084 Note: \( u_3 \) not shown in Fig. 5, it is the vertical component.

p. 1085 Change Eq. 44 to \([F]_i = [b]_i [x]_i = [b]_i [(v) [a]]T(\Delta) = [P]_i (\Delta)\)

p. 1086 1. Change Eq. 47 to \( Q_i = m r^2 (i) \Delta_i \)
2. Change Eq. 49 to \(-[S] (\Delta) = m[I'] [\Delta']\)
3. Change Eq. 50 to \([S] - \lambda n[R] [A]_n = 0\)

p. 1091 (Caution) It has been said and not yet verified that the characteristic vectors in Table 4(g) are incorrect.

p. 1095 Following the Eq. for \( S'_{45} \) change \( B'_{46} \) to \( S'_{46} \).

2. Discussions of the above paper were published as follows:
3. Discussions or other papers which add to the above.
INTRODUCTION

A general method of analysis by direct stiffness of three-dimensional pile foundations for static loading or dynamic response is given herein. The pile foundation consists of a group of piling placed into the soil topped with a reinforced concrete cap. Loads to the cap and the weight of the cap are borne by the piling to the soil. Determinations of deflections and individual pile loads are required by the designer. Adequate representation of soil-pile interaction is necessary.

The general formulation of linear structural systems by direct stiffness has been adequately presented by Lind (18), Spillers (22), Fenves and Bralin (12), and Fenves (12), and a related type of problem has been presented by Weaver and Nelson (27). Applied to pile foundations, the behavior of individual piles in their local coordinates is related to the system through the global coordinates to obtain the stiffness of the whole foundation. Calculations may be made either through matrix mathematics methods using the matrices defined in the analysis or by means of formulas for the influence coefficients derived from the analysis.

Asplund (2,3), and Vetter (26), in excellent reviews of historical developments in the analysis of pile foundations, credit P. Gullander with an early contribution in 1902. Asplund notes an early series of papers culminating with that of C. Nekkentved in 1924. A recent series of contributions from about 1945 has attempted to update the earlier works by modernizing the mathematical formulation, including soil-pile interaction, and by imposing less restrictions on the problem properties. Among the latest, Hrennikoff (15) published a planar analysis which included hinged or fixed connections at the cap, use of batter piles, and lateral resistance by the soil to pile movement. First ap-

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Note.—Discussion open until October 1, 1968. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 94, No. ST5, May, 1968. Manuscript was submitted for review for possible publication on July 14, 1967.

\(^{1}\) Assoe. Prof. of Civ. Engrg., Univ. of Wisconsin, Madison, Wis.

\(^{2}\) Numerals in parentheses refer to corresponding items in Appendix III.—References.
proximation methods useful in design are also described. Aschenbrenner's (1) solution is extended to three dimensions but is limited to hinged connections. Hrennikoff (15) and Aschenbrenner (1) presented derived formulas; Rising, Roth, and Anderson (21) included matrix formulation as well, although their problem is severely limited to hinged connections and no lateral soil resistance. Turzynski (25) considered matrix methods but is more restrictive than Rising, et al. In that, in addition to their limitations, only planar loading is considered. Juhl (16) used matrix methods but, again, is more restrictive than Rising, et al. In that, in addition to the limitations already existing, only vertical piling are considered. Asplund (2, 3) uses matrix methods in a fairly general formulation but sacrifices usefulness by introducing complicating fictitious devices for operating at the inflection points of the piles.

The method presented herein assumes a rigid piling cap and elastic behavior of the system. Otherwise, there may be fixed or hinged connections at the cap; piles with different bending stiffness about their principal axes ($f_x, f_y$, a feature formerly disregarded); any degree of linear torsional or axial stiffness, or lateral soil resistance, including zero; any position and batten of piles; or piles of different sizes, materials, and end conditions in the same foundation. In addition, the designer may, at his discretion, consider the piling for calculation of their lateral resistance as a semi-infinite beam on an elastic foundation, as a cantilever fixed at some depth, or as a cantilever for some depth adjoined to a semi-infinite beam on an elastic foundation. He may determine the elastic constants by field tests. Although matrix methods are used and electronic computation leads itself most readily to obtain a solution, formulas for the stiffness influence coefficients and for obtaining the pile forces are given herein.

Whenever pile foundations are used for machine foundations or subjected to shock or buffeting such as in the case of ocean structures, the dynamic properties are of interest. The frequency equation, used to obtain the principal frequencies and modes, is derived and examined herein. Finally, illustrative problems will be worked out in detail to show the application of the method and a limited study made to compare effects of the variation of parameters and their relative significance.

**LATERAL LOADING OF PILES**

The lateral load capacity of piling has been the subject of considerable interest in foundation design (11, 23). For analytical purposes a model representative of the soil-pile interaction is required. The most primitive model simply disregards lateral capacity and assumes all pile forces to be axial, thereby requiring batter piles to pick up lateral load components. Modeling the pile as a cantilever beam fixed at a particular depth has been advocated by Davisson and Robinson (10); soil properties enter into the determination of effective length.

A semi-infinite beam on a linear spring foundation (elastic foundation) serves as a most useful model, and has lead to an entire family of specialized models which are more nearly correct for real soils. Terzaghi (24) suggested an effective spring stiffness, $k_s$, constant with depth for cohesive soils and linearly increasing with depth from zero for granular soils, $k_s$z. Furthermore, he proposed how $k_s$ may be determined from a plate bearing test on a
1-ft square horizontal plate and suitably modified for width, length, or depth effects. Matlock and Reese (19) presented generalized solutions for a nonlinear soil, noting that the soil stiffness increased with depth because of an increase in soil strength and smaller pile deflections. They concluded that there was not much sensitivity of the stress resultant in the pile to variation in soil stiffness, and they also described rigid pile or pole behavior. Pole behavior has been discussed by Davison and Prakash (9) and by Broms (5, 6, 7), who note that when the wave length $\beta l$ is less than about 2 the entire pile moves with rigid body motion in addition to bending and can no longer be approximated as a semi-infinitely long beam. Prakash (20) studied the influence of group behavior and concluded that for spacing less than three pile diameters the soil stiffness should be decreased and that spacing greater than eight diameters has no effect. Broms (6) says to use $2/3$ of $k_c$ in sands for group effect or under vibration or repetition of load, and to let $k_c = 0$ for a depth 1.5 times the pile diameter in cohesive soils (5). Davison and Gill (6) studied layered systems and concluded that the surface layer is of major importance. Comprehensive studies by Broms (5, 6, 7) and by Wilson and Hills (28) review the state of the art, advance design aids, delineate broad limits of applicability, and discuss shortcomings.

Although three alternate methods of determining pile stiffness constants are offered herein, the analysis is quite general, and it is the designer's prerogative to select values of his choice.

**ISOLATED PILE ACTION**

The forces and displacements at the head of a single piling are shown in Fig. 1 in an orthogonal right triad in which axes 1 and 2 are principal axes of inertia.

![Diagram of forces and displacements](image)

**FIG. 1.—PILE \( i \)**

And axis 3 coincides with the longitudinal axis of the piling. There are six components of generalized displacements \( \{x\} \), of which three are linear and three rotational. In addition, the six components of generalized forces \( \{F\} \) are three forces and three moments. These are related by individual pile stiffness influence coefficients, hereafter called elastic pile constants, such that

\[
\{F\}_i = \{b\}_i \{x\}_i \quad \text{..................} \quad (1)
\]
in which the subscript $i$ identifies a particular piling; and the elements of the elastic pile constants matrix $[b]$ are dependent on the axial, flexural, and torsional stiffnesses of the pile as well as its interaction with the surrounding soil media and the boundary conditions.

The elastic pile constants may be obtained directly from a load test, indirectly through soil tests and their interpretation for use in an analytical formulation, or empirically. Columns of the flexibility matrix $[b]^{-1}$ can be established directly from a load test since each column element, $b_{ij}$, is the displacement at $i$ due to a unit force at $j$. The empirical approach requires judgment and experience. The analytical approach must be supplemented and modified with field observations, but it appears to be the most useful. Three possible analytical formulations for the pile constants are offered, as given below.

The pile may be analyzed as a semi-infinite beam on an elastic foundation for flexure, as a modified compression block for axial deformation, and as a modified shaft in torsion. Thus, the elastic pile constants matrix would be defined as follows:

$$
[b] = \begin{bmatrix}
\delta_{11} & 0 & 0 & 0 & \delta_{14} & 0 \\
0 & \delta_{22} & 0 & \delta_{24} & 0 & 0 \\
0 & 0 & \delta_{33} & 0 & 0 & 0 \\
0 & \delta_{34} & 0 & \delta_{44} & 0 & 0 \\
\delta_{51} & 0 & 0 & 0 & \delta_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \delta_{66}
\end{bmatrix}
$$

(2)

in which, with dimensions of units given as $L$ = length, $F$ = force, and $R$ = radians,

$$
\frac{F}{L} \delta_{11} = (1 + \delta) \kappa_1 \beta_1^i 
$$

(3)

$$
\frac{F}{L} \delta_{22} = (1 + \delta) \kappa_2 \beta_2^i 
$$

(4)

$$
\frac{F}{L} \delta_{33} = k_L \left( \frac{AE}{L} \right) 
$$

(5)

$$
\frac{F}{} \delta_{44} = \delta \kappa_1 
$$

(6)

$$
\frac{F}{R} \delta_{55} = \delta \kappa_2 
$$

(7)

$$
\frac{F}{} \delta_{66} = \delta \kappa_T \left( \frac{JC}{L} \right) 
$$

(8)

$$
\frac{F}{} \delta_{15} = \delta_{51} = \delta \kappa_1 \beta_1 
$$

(9)

$$
\frac{F}{} \delta_{24} = \delta_{42} = -\delta \kappa_2 \beta_2 
$$

(10)

$$
\frac{1}{L} \beta_1^i = \frac{k_D}{AEI} 
$$

(11)
\[ LF \quad \kappa_i = 2 \beta_i \frac{EI}{t} \]  

(12)

in which \( k_T \) and \( k_L \) = combined shape and participation factors in torsion and compression respectively; \( k_T \) = the modulus of subgrade reaction(24); \( D_i \) = the projected average width of the pile in the direction of bending; \( EI_1, JG/L, \) and \( AE/L \) = the flexural, torsional, and axial rigidities, respectively; and the end conditions are defined by \( \delta = 0 \) for a hinged connection and \( \delta = 1 \) for the fixed condition. Note that for the hinged condition the matrix sizes are reduced. To achieve a fixed condition firm imbedment of the piling within the cap and mechanical aids are required. The factors \( k_T \) and \( k_L \) may be taken as 1 for end bearing piles or 2 for friction piles. The modulus of subgrade reaction \( k_g \) should be determined from plate bearing tests, but in a broad sense it is given as follows(24):

For highly compressible silt, clay or organic material, \( k_g = 0 \) pci to 170 pci; for low to medium silt or clay, \( k_g = 150 \) pci to 225 pci; for sand, fine or poorly graded, \( k_g = 200 \) pci to 300 pci and for clayey or well graded sand, fine gravel, \( k_g = 250 \) pci to 450 pci.

At the discretion of the designer, the elastic pile constants in bending may be derived using as a model a cantilever beam fixed at an arbitrary depth as follows(10):

\[ b_{11} = (1 + \delta) \frac{3EI_1}{L^3} \]  

(13)

\[ b_{22} = (1 + \delta) \frac{3EI_2}{L^3} \]  

(14)

\[ b_{33} = \delta \left( \frac{AE_1}{L} \right) \]  

(15)

\[ b_{44} = \delta \left( \frac{AE_2}{L} \right) \]  

(16)

\[ b_{15} = b_{11} = \delta \left( \frac{6EI_1}{L^2} \right) \]  

(17)

\[ b_{24} = b_{22} = -\delta \left( \frac{6EI_2}{L^2} \right) \]  

(18)

The drawback to this model is in defining the length, \( t \), which is a function of the soil properties unless obviously justified under particular circumstances. Another possible alternative occurs especially in elevated platforms such as those used offshore where the piling extend above soil level. The elastic pile constants for bending may be derived by considering the pile as a cantilever beam of two regimes: One free of lateral soil restraint, and the second as a semi-infinite beam on a spring foundation. Note that stability should be checked for each case(10,17). The elastic pile constants for this case, in which \( f \) is the laterally unsupported length, are

\[ b_{11} = 3 [1 + \delta (1 - 2 \beta_3 t)] \beta_3^2 \kappa_1 t_1 \]  

(19)

\[ b_{22} = 3 [1 + \delta (1 - 2 \beta_3 t)] \beta_3^2 \kappa_2 t_2 \]  

(20)

\[ b_{15} = \delta \kappa_1 (3 + 6 \beta_3 t + 6 \beta_1 t^2 + 2 \beta_3^2 t^3) t_1 \]  

(21)

\[ b_{24} = \delta \kappa_2 (3 + 6 \beta_3 t + 6 \beta_2 t^2 + 2 \beta_3^2 t^3) t_2 \]  

(22)
\[ b_{n1} = b_{n1} = 3 \beta \beta_1 \gamma \gamma_1 (1 + 2 \beta_1 f + \beta_2 f^2) t_1 \]  
(23)

\[ b_{n3} = b_{n4} = -3 \beta \beta_2 \gamma \gamma_2 (1 + 2 \beta_2 f + \beta_1 f^2) t_3 \]  
(24)

\[ t_i = \frac{1}{3 + 6 \beta f + 6 \beta_1 f^2 + (1 + \delta) 2 \beta_2 f^2 + 2 \beta_1 f^2} \]  
(25)

In any case the elements of \( b \), Eq. 2, are defined, following the designer's discretion, by Eqs. 3 through 12 or Eqs. 13 through 18 plus Eq. 5 and Eq. 8, or by Eqs. 19 through 25 plus Eq. 5 and Eq. 8.

A pile may be located at a rotated position with respect to the established coordinate system of the foundation and battered. Its spatial position is fully defined by the clockwise angle \( \alpha_i \) to the direction of batter and the batter slope \( \beta_i \), as shown in Fig. 2(b). The components of force or displacement orthogonal

![Diagram](image)

**FIG. 2.—ORIENTATION OF PILE WITH RESPECT TO FOUNDATION**

to the rotated axes are found by transformation. The transformation matrix \( [a]_i \) for pile \( i \) [see Fig. 2(a)] is

\[
[a]_i = \begin{pmatrix}
[a']_i \\
0
\end{pmatrix}
\]  
(26)

where \( [a']_i = \begin{pmatrix}
\cos \gamma & \cos \alpha & -\sin \gamma & \cos \alpha \\
\cos \gamma & \sin \alpha & \sin \gamma & \cos \alpha \\
-\sin \alpha & 0 & \cos \gamma & 0
\end{pmatrix} \]  
(27)

and \( \gamma_i = \arccot \beta_i \)  
(28)

is the angle with respect to axis \( 3' \). The ranges are

\[ 0^\circ \leq \alpha \leq 360^\circ \]  
(29)

\[ 0^\circ \leq \gamma < 90^\circ \]  
(30)

on the location angles, however, the practical upper limit on \( \gamma \) is much lower than \( 90^\circ \) since \( 90^\circ \) represents a horizontal pile. Thus the equations

\[ \{F\}'_i = [a]_i \{F\}_i \]  
(31)

\[ \{x\}'_i = [a]_i^T \{x\}_i \]  
(32)
in which \([a]^T\) is the transpose of \([a]\), give the force and displacement transformation relationships, respectively. Substitution of Eqs. 1 and 32 into Eq. 31 yields

\[
\{F_i\} = [a]_i [b]_i [a]^T \{x_i\} 
\]

which is the relationship of the pile's force to its deflections in an orthogonal coordinate system (local) parallel to the chosen (global) axes of the foundation.

GROUP ACTION

Pile \(i\) may be located in the foundation as shown in Fig. 3 where the primed axes (local) are parallel to a set of axes \(\{U\}\) (global) defined for the foundation.

![Fig. 3.—Position of Pile in Cap](image)

The \(U_3\) axis would be normal to the surface of the earth. The origin and orientation of \(U_1\), besides being in the midplane of the piling cap, may be chosen at will. Usually, an origin at the centroid of the piling group and \(U_1\) in a line of symmetry, should one exist, would simplify calculations. However, other considerations, such as known foundation load components in a particular position and orientation, may dictate choice. Each pile \(i\) is located in the midplane of the pile cap by its coordinates \(\{u\}_i\) in which \(u_3 = 0\).

The foundation loads \(\{Q\}\) and displacements \(\{\Delta\}\) are located with respect to the foundation axes \(\{U\}\) as shown in Fig. 4. If the origin is through the cap centroid \(Q_3\) may include the weight of the foundation. Otherwise, components will enter into \(Q_4\) and \(Q_5\) as moments.

The forces \(\{F_i\}\) due to pile \(i\) on the piling cap at \(\{u\}_i\) are in equilibrium with a set of forces \(\{q\}_i\) at the coordinate center of the cap as shown in Fig.
5. Equilibrium yields

\[ \{q\}_i = [c]_i \{F^r\}_i \]  \hspace{1cm} (34)

in which

\[ [c]_i = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -\mu_2 & \mu_4 & \delta & 0 \\
\mu_3 & 0 & -\mu_1 & 0 & 0 \\
-\mu_3 & \mu_1 & 0 & 0 & \delta \\
\end{bmatrix} \]  \hspace{1cm} (35)

is the statics matrix and \( \delta \) is as previously defined.

\[ \text{FIG. 5.—EQUILIBRIUM OF PILE } i \]

Considering that the piling cap is assumed rigid, its deflection can be related to the piling to give

\[ \{x\}_i = [c]_i^T \{\Delta\} \]  \hspace{1cm} (36)

in which \([c]_i^T\) is the transpose of the statics matrix, \([c]_i\), of pile \(i\).

The foundation load is distributed to each piling so that

\[ \{Q\} = \sum_{i=1}^{n} \{q\}_i \]  \hspace{1cm} (37)

in which \(n\) is the total number of piling.

The relationship between the foundation loads and the pile cap deflections can be written

\[ \{Q\} = [S] \{\Delta\} \]  \hspace{1cm} (38)

in which the \(S_{ij}\) are stiffness influence coefficients for the foundation as a whole, \(S_{ij}\) being the force in direction \(i\) at the origin required for a unit deflection of the rigid cap in direction \(j\). The \([S]\) matrix may be found by substitution of Eqs. 33 and 36 into Eq. 34 to introduce the contribution of each pile toward \(S_{ij}\). Thus,

\[ \{q\}_i = [S^r]_i \{\Delta\} \]  \hspace{1cm} (39)
in which \[ [s^t]_i = [a]_i [b]_i [a]_i^T [c]_i^T \tag{40} \]

and, therefore,

\[ [s] = \sum_{i=1}^{n} [s^t]_i \tag{41} \]

Matrix operations are easily followed for hand computations or machine programming. In addition, formulas for the influence coefficients \( s_{ij} \) are given for the general case, the case of symmetrical piling in which \( f_x = f_y \), the case of vertical piling only, and the general case with hinged connections in Appendix I.

**PILING LOADS**

If piling cap deflections are specified, the allowable foundation loads are given by Eq. 36. The converse problem to determine the foundation movements from specified loads can be found from

\[ \{\Delta\} = [s]^{-1} \{q\} \tag{42} \]

in which the inverse \([s]^{-1}\) = the flexibility matrix for the foundation as a whole.

The deflection of pile \( i \) is determined by substitution of Eq. 36 into Eq. 32 to yield

\[ \{x\}_i = [a]_i^T [c]_i^T \{\Delta\} \tag{43} \]

which is equivalent to

\[ \{x\}_i = ([c]_i [a]_i)^T \{\Delta\} = [p]_i \{\Delta\} \tag{44} \]

Formulas for the \( p_{ij} \) are given in Appendix II in general form. Finally, the forces allotted to each piling from either imposed loading or displacement follow from Eq. 1.

**DYNAMICS OF PILE FOUNDATIONS**

With reference to the mass center, the equation of motion for translation in the \( U_i \) direction is

\[ Q_i = m \ddot{x}_i \tag{45} \]

in which \( m \) = the mass of the vibrating system and \( \ddot{x} \) = acceleration. A portion of the soil held within the pile group, probably dependent on relative amplitude of vibration, as well as the cap, upper portion of the pile, and attached ap-
purtenances, constitute the mass. In general, the translatory equations of motion are

\[ Q_i = m \ddot{A}_i; \quad i = 1, 2, 3 \]  \hspace{1cm} (46)

In rotation, reference to the mass center is preferred so that the equations of motion may be written in their simplest form as

\[ Q_i = m r_i^2 \Delta_i; \quad i = 4, 5, 6 \]  \hspace{1cm} (47)

in which \( r_i \) is the radius of gyration of the mass with respect to the \( i \) axis. For a rectangular parallelepiped of dimensions \( l_i \) in the 1-direction by \( l_2 \) by \( l_3 \), the radii of gyration are

\[
\begin{align*}
  r_1^2 &= \frac{1}{12} (l_2^4 + l_3^4) \\
  r_2^2 &= \frac{1}{12} (l_1^4 + l_3^4) \\
  r_3^2 &= \frac{1}{12} (l_1^4 + l_2^4)
\end{align*}
\]  \hspace{1cm} (48)

Radii of gyration for other shapes are readily available. Eqs. 46 and 47 can be written compactly as

\[
[S] [A] = m [I'] \{ \Delta \} \]  \hspace{1cm} (49)

in which \([I']\) in a diagonal matrix with \( I_{11} = I_{22} = I_{33} = 1 \) and \( I_{14} = r_i^2 \) for \( i = 4, 5 \) or 6.

By assuming harmonic motion for the elastic system to which the applied forces have the same time variation, a frequency equation of the form

\[
[I] - \lambda_m [I'] \{ A \}_m = 0 \]  \hspace{1cm} (50)

can be derived (4), in which \( \{ A \}_m \) = the characteristic vector of the \( m \)-th node, \( \lambda_m \) = the characteristic value, and \([I]\) = the identity matrix (a diagonal matrix of unity). The principal frequencies are

\[
\omega_m = \sqrt{\lambda_m/m} \]  \hspace{1cm} (51)

and

\[
[S] = [I']^{-1} [S] \]  \hspace{1cm} (52)

which can be shown to be composed of elements

\[
\dot{S}_{ij} = S_{ij} \quad i = 1, 2, 3; \quad j = 1, \ldots, 6 \]  \hspace{1cm} (53)

\[
\ddot{S}_{ij} = \frac{S_{ij}}{r_i^2} \dot{r}_{i(f)} \quad i = 4, 5, 6; \quad j = 1, \ldots, 6 \]  \hspace{1cm} (54)

Solution of Eq. 50, which has six degrees of freedom, is best done numer-
ically such as by Stodola's Method(4) or Householder's Method(14), whether calculated by hand or computer.

Once principal frequencies are known, destructive resonance conditions may be avoided by changing the mass of the structure or through alterations in machine design. For the design of machine foundations of known excitation frequencies, a number of possible foundations may be analyzed as described above for an acceptable response.

ILLUSTRATIVE PROBLEMS

Two problems are given below. The first is a three-pile footing designed only to illustrate the method and its flexibility. It is easily hand checked. The

![Diagram of three-pile foundation](image)

FIG. 6.—PROBLEM NO. 1, THREE-PILE FOUNDATION

<table>
<thead>
<tr>
<th>Pile</th>
<th>$\alpha$</th>
<th>$h$</th>
<th>$w_1$, in inches</th>
<th>$w_2$, in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30°</td>
<td>3</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>0°</td>
<td>Vertical</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>300°</td>
<td>4</td>
<td>-20</td>
<td>-20</td>
</tr>
</tbody>
</table>

second problem is treated more broadly to examine the relative effect of various parameters. A computer program was written in FORTRAN and prob-
TABLE 2.—INFLUENCE COEFFICIENTS FOR PROBLEM NUMBER 1

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Pile Number</th>
<th>Foundation</th>
<th>Pile Number</th>
<th>Foundation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$S_{1}$</td>
<td>58.27</td>
<td>26.50</td>
<td>18.86</td>
<td>120.94</td>
</tr>
<tr>
<td>$S_{2}$</td>
<td>0</td>
<td>-10.04</td>
<td>10.56</td>
<td>0</td>
</tr>
<tr>
<td>$S_{3}$</td>
<td>102.78</td>
<td>0</td>
<td>-6.28</td>
<td>149.32</td>
</tr>
<tr>
<td>$S_{4}$</td>
<td>2525.7</td>
<td>0</td>
<td>-930.3</td>
<td>1724.2</td>
</tr>
<tr>
<td>$S_{5}$</td>
<td>-1043.3</td>
<td>0</td>
<td>1043.3</td>
<td>-21.52</td>
</tr>
<tr>
<td>$S_{6}$</td>
<td>-371.0</td>
<td>0</td>
<td>371.0</td>
<td>228.4</td>
</tr>
<tr>
<td>$S_{7}$</td>
<td>58.29</td>
<td>28.50</td>
<td>45.35</td>
<td>122.34</td>
</tr>
<tr>
<td>$S_{8}$</td>
<td>59.24</td>
<td>0</td>
<td>-80.62</td>
<td>21.27</td>
</tr>
<tr>
<td>$S_{9}$</td>
<td>1.186</td>
<td>0</td>
<td>1.212</td>
<td>2.759</td>
</tr>
<tr>
<td>$S_{10}$</td>
<td>-1.700.3</td>
<td>0</td>
<td>-1.512.3</td>
<td>-1.252.6</td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>909.0</td>
<td>0</td>
<td>-1.120.4</td>
<td>-3.11.5</td>
</tr>
<tr>
<td>$S_{12}$</td>
<td>244.5</td>
<td>424.1</td>
<td>400.8</td>
<td>1.209.5</td>
</tr>
<tr>
<td>$S_{13}$</td>
<td>7,491</td>
<td>0</td>
<td>-8,017.7</td>
<td>322.6</td>
</tr>
<tr>
<td>$S_{14}$</td>
<td>11,537</td>
<td>0</td>
<td>-11,910.2</td>
<td>-11,993</td>
</tr>
<tr>
<td>$S_{15}$</td>
<td>-275.4</td>
<td>0</td>
<td>2,543.5</td>
<td>2,847.8</td>
</tr>
<tr>
<td>$S_{16}$</td>
<td>159,820</td>
<td>0</td>
<td>159,820</td>
<td>159,820</td>
</tr>
<tr>
<td>$S_{17}$</td>
<td>-320,120</td>
<td>0</td>
<td>-320,120</td>
<td>-239,070</td>
</tr>
<tr>
<td>$S_{18}$</td>
<td>-3,510</td>
<td>0</td>
<td>3,510</td>
<td>5,833.3</td>
</tr>
<tr>
<td>$S_{19}$</td>
<td>345,500</td>
<td>0</td>
<td>345,500</td>
<td>504,440</td>
</tr>
<tr>
<td>$S_{20}$</td>
<td>8,263</td>
<td>0</td>
<td>50,863</td>
<td>59,158</td>
</tr>
<tr>
<td>$S_{21}$</td>
<td>37,180</td>
<td>0</td>
<td>49,170</td>
<td>77,430</td>
</tr>
</tbody>
</table>

FIG. 7.—PROBLEM NO. 2, NINE-PILE FOUNDATION
Problem No. 1.—Data for the three-pile foundation shown in Fig. 6 are given in Table 1. In the problem, $E = 1500$ ksl, $G = 800$ ksl, $D = 12$-in. diam. round, $L = 400$-in. long piles, $k_T = 1.0$, $k_L = 1.0$, $k_s = 0.1$ kcl, and other values are given with the results. The elastic coefficients for the hinged condition are, assuming lateral resistance as a beam on an elastic foundation; $b_{31} = b_{32} = 28.5$ kips per in., and $b_{33} = 424$ kips per in.; for the fixed condition, $b_{31} = b_{32} = 57.0$ kips per in., $b_{33} = 1353.6$ in.-kips per in., $b_{43} = -1353.6$ in.-kips per in., $b_{44} = b_{55} = 64,291.0$ in.-kips per radian, and $b_{56} = 4072.0$ in.-kips per radian. The stiffness influence coefficients are given in Table 2 for one- and three-pile foundations.

Problem No. 2.—Data for the nine-pile foundation in Fig. 7 are given in Table 3. Parameters were varied to study effects of the torsion constant $b_{56}$, end conditions, and lateral restraint of soil on deflections, distribution of foundation load to piling, and dynamic response. Table 4 gives a complete analysis for the fixed condition with $k_s = 0.1$ kcl. It may be noted that the torsion-moments to each pile are not significant, which indicates that rotation

<table>
<thead>
<tr>
<th>Pile</th>
<th>$a$, in degrees</th>
<th>$k$</th>
<th>$u_1$, in inches</th>
<th>$u_6$, in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>180</td>
<td>4</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>150</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>135</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>180</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>225</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>210</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>200</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*Friction piling: $k_T = 2$ (except as noted), $k_L = 2$, $D = 10.75$-in. Steel XH (ASA 60 0.50-in. wall), $E = 30,000$ ksl, $G = 12,000$ ksl, $A = 16.10$ sq in., $I = 211.9$ in.$^4$, $L = 120$ ft, and $m = 0.335$ kip-sec$^2$ per in.

Table 3.—Data for Problem No. 2

In the plane of the foundation is mainly restrained by shear forces to the piling. The largest axial force is $332.9$ kips on Pile No. 5 and the greatest shear is $59.5$ kips on Pile No. 6. The frequencies are in two groups of three very close values. Letting $b_{56} = 0$ (piling with no torsional restraint) in the prior problem produces nearly the same solution with the following differences: $S_{56} = 864.820$ in.-kips per radian, $S_{56} = 12,810,700$ in.-kips per radian, $S_{46} = 2,988,700$ in.-kips per radian, $\Delta_e = 0.001259$ radians, and $F_1 = 0$ for all piling. Since the individual pile forces and the frequency response are virtually unchanged, torsional restraint of the individual piling appears to be of little significance.

Relative effects of the soil modulus $k_s$ on the nine-pile foundation with fixed or hinged connections at the cap are given in Table 5. The maximum pile loads, foundation deflection, and first and second principal frequencies are listed. The axial components of the pile forces are listed in Table 6 for four different values of $k_s$ for the foundation with $\delta = 1$ and one with $\delta = 0$. Negative denotes
### TABLE 4.—ANALYSIS OF 9-PILE FIXED FOUNDATION

#### (a) Nonzero Elements of [h] Matrix
- \( h_{11} = h_{22} = 74,989 \text{ kips per in.} \)
- \( h_{12} = h_{21} = 74,989 \text{ kips per in.} \)
- \( h_{33} = 670.83 \text{ kips per in.} \)
- \( h_{34} = h_{43} = 182,308 \text{ in.-kips per radian} \)
- \( h_{35} = h_{53} = 7,063.3 \text{ in.-kips per radian} \)
- \( h_{36} = h_{63} = 2,614.1 \text{ in.-kips per in.} \)

#### (b) Stiffness Influence Coefficient Matrix \([S]\)

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( S_{11} = 911.32 \text{ kips per in.} )</td>
<td></td>
<td>( S_{22} = 5,554.4 \text{ kips per in.} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( S_{22} = S_{33} = S_{44} = 0 )</td>
<td></td>
<td>( S_{55} = S_{66} = S_{77} = 0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( S_{12} = 635.85 \text{ kips per in.} )</td>
<td></td>
<td>( S_{34} = -11,760.5 \text{ in.-kips per in.} )</td>
<td></td>
<td>( S_{45} = 8,398,000 \text{ in.-kips per radian} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( S_{13} = -6,036.7 \text{ in.-kips per in.} )</td>
<td></td>
<td>( S_{46} = 841.22 \text{ kips per in.} )</td>
<td>( S_{56} = 5,554.4 \text{ kips per in.} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( S_{23} = 0 )</td>
<td>( S_{55} = 0 )</td>
<td>( S_{66} = 0 )</td>
<td></td>
<td>( S_{77} = 0 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( S_{35} = 857.280 \text{ in.-kips per radian} )</td>
<td></td>
<td>( S_{47} = 6,887.9 \text{ in.-kips per in.} )</td>
<td></td>
<td>( S_{57} = 12,412,600 \text{ in.-kips per radian} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( S_{36} = 1,772.36 \text{ in.-kips per in.} )</td>
<td></td>
<td>( S_{58} = 3,047,500 \text{ in.-kips per radian} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### (c) Foundation Deflections
- \( \Delta_1 = 0.488 \text{ in.} \)
- \( \Delta_2 = 0.1063 \text{ in.} \)
- \( \Delta_3 = 0.030 \text{ in.} \)
- \( \Delta_4 = 0.001215 \text{ radians} \)
- \( \Delta_5 = 0.00438 \text{ radians} \)
- \( \Delta_6 = 0.001237 \text{ radians} \)

#### (d) Pile Loads

<table>
<thead>
<tr>
<th>Pile Number</th>
<th>( P_{11} ) in kips</th>
<th>( P_{12} ) in kips</th>
<th>( P_{21} ) in kips</th>
<th>( P_{22} ) in inch-kips</th>
<th>( P_{31} ) in kips</th>
<th>( P_{32} ) in inch-kips</th>
<th>( P_{33} ) in inch-kips</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-52.4</td>
<td>-4.7</td>
<td>118.6</td>
<td>-27.7</td>
<td>-2,225.7</td>
<td>4.4</td>
<td></td>
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<tr>
<td>2</td>
<td>48.0</td>
<td>9.8</td>
<td>63.9</td>
<td>-231.0</td>
<td>2,570.9</td>
<td>8.7</td>
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<td>3</td>
<td>25.7</td>
<td>-31.7</td>
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<td>997.2</td>
<td>18.1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-41.2</td>
<td>-25.8</td>
<td>179.0</td>
<td>-972.4</td>
<td>-1,838.2</td>
<td>10.4</td>
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</tr>
<tr>
<td>5</td>
<td>-44.9</td>
<td>-25.7</td>
<td>332.9</td>
<td>1,192.6</td>
<td>-1,524.1</td>
<td>12.3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-59.5</td>
<td>1.1</td>
<td>256.4</td>
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<td>-2,473.0</td>
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</tr>
<tr>
<td>7</td>
<td>-47.2</td>
<td>37.3</td>
<td>240.5</td>
<td>-1,578.5</td>
<td>-1,847.2</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-51.1</td>
<td>21.8</td>
<td>79.4</td>
<td>-1,074.9</td>
<td>-2,086.8</td>
<td>2.9</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>50.0</td>
<td>50.4</td>
<td>52.6</td>
<td>-2,068.3</td>
<td>850.7</td>
<td>1.2</td>
<td></td>
</tr>
</tbody>
</table>

#### (e) Pile Deflections

<table>
<thead>
<tr>
<th>Pile Number</th>
<th>( x_{11} ) in inches</th>
<th>( x_{12} ) in inches</th>
<th>( x_{21} ) in inches</th>
<th>( x_{22} ) in inches</th>
<th>( x_{31} ) in radians</th>
<th>( x_{32} ) in radians</th>
<th>( x_{33} ) in radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.548</td>
<td>-0.1137</td>
<td>0.1767</td>
<td>-0.001479</td>
<td>-0.00346</td>
<td>0.00091</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.488</td>
<td>0.1731</td>
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<td>-0.040</td>
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<td>0.00079</td>
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<td>7</td>
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<td>0.259</td>
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<td>-0.00222</td>
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<td>8</td>
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<td>0.114</td>
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<td>-0.00016</td>
<td>0.00042</td>
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<td>0.080</td>
<td>-0.000339</td>
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<td>0.000173</td>
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</tbody>
</table>
TABLE 4.—CONTINUED

(f) Frequencies, in radians per second

| \( \omega_0 = 77.9 \) | \( \omega_1 = 86.4 \) | \( \omega_2 = 218.8 \) |
| \( \omega_3 = 78.5 \) | \( \omega_4 = 205.0 \) | \( \omega_5 = 238.2 \) |

(g) Characteristic Vectors

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<th>Vector</th>
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<th>Component 2 ( A_2 )</th>
<th>Component 3 ( A_3 )</th>
<th>Component 4 ( A_4 )</th>
<th>Component 5 ( A_5 )</th>
<th>Component 6 ( A_6 )</th>
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<td>-0.1220</td>
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TABLE 5.—EFFECTS OF VARIATION OF SOIL MODULUS ON FIXED AND HINGED NINE-PILE FOUNDATION

| \( \delta \) | k_{ph} in kips per cubic inch | Maximum Loads (Pile Number) | \( \omega_1 \) in radians per second | \( \omega_2 \) in radians per second |
|-------------|-------------------------------|-----------------------------|-----------------------------------|
| \( k_{ph} \) | in kips                       | \( Q_1 \) or \( Q_2 \) in kips | \( Q_3 \) in inch kips            |                                   |
| 0           | 0                             | 1,154.3(3)                 | 0                                 | 11.030                           | 10.29                    | 17.85                    |
| 0           | 0.01                          | 602.2(7)                   | 35.9(6)                           | 0                                 | 4.978                    | 23.60                    | 27.94                    |
| 0           | 0.1                           | 435.7(5)                   | 53.7(6)                           | 0                                 | 1.369                    | 51.04                    | 63.33                    |
| 0           | 1.0                           | 377.6(5)                   | 63.0(6)                           | 0                                 | 0.224                    | 118.03                   | 118.57                   |
| 1           | 0                             | 1,101.3(3)^a               | 0                                 | 11.025                           | 10.76                    | 19.15                    |
| 1           | 0.01                          | 402.8(5)                   | 41.8(6)                           | 0                                 | 2.265                    | 39.90                    | 40.20                    |
| 1           | 0.1                           | 332.9(5)                   | 59.5(6)                           | 0                                 | 0.488                    | 77.90                    | 78.55                    |
| 1           | 1.0                           | 313.3(5)                   | 91.8(6)                           | 2.310(6)                         | 0.022                    | 156.24                   | 160.48                   |

^a Torsional restraint effects show differences in these cases.

A tensile force on the piling. The relative effect on distribution of foundation loads to the piling are similar [when \( \delta = 0 \) (hinged connections)] to the fixed case except that there are no moment components.

PROBLEM SOLUTIONS

An analysis should be verified by a statics or deformation check. Satisfaction of equilibrium is especially useful for checking, since it reinforces a physical feeling for the problem in the designer. Very slender piling, or that essentially free from lateral restraint, should be checked for stability.

In Table 2 each column marked "Pile Number" represents a single pile foundation with its respective pile. The column labeled "Foundation" gives the stiffness influence coefficients for the three-pile foundation. Both the hinged and fixed cases are represented. The sum of the stiffness influence coefficients of each pile with respect to a common coordinate center equals the coefficient for the group. It may be noted that there is a difference between the two cases, some coefficients being considerably different.

A complete analysis of the nine-pile foundation described in Table 3 with
A study of Tables 5 and 6 reveals that for the problem studied both the end conditions and soil modulus have considerable effect on the analysis. The fixed condition increased the frequency response by as much as 63% over that of the hinged condition. This decreased the maximum pile axial force by as much as a third, while increasing the shear force slightly and adding the end moments. Foundation deflections were decreased by up to 90%. The reduction in maximum pile forces was accompanied by a more even distribution of the foundation loads to the piles as can be seen in Table 6 comparing the columns with $k_s = 0.01$ kcl for $\delta = 0$ and $\delta = 1$.

Taking account of only a minimal soil modulus of 10 pct (a very soft soil) over no lateral restraint has very considerable effects. Pile loads are distributed more evenly (Table 6), with a decrease of 63% in the maximum pile force. Deflections are decreased by up to 80%. Frequencies are increased by almost three times. Neglecting soil-pile interaction appears to give erroneous results in this case, since taking account of even a minimal soil modulus affects gross changes in the solution.

The magnitude of $k_s$ is an elusive figure assumed to be a constant herein, but it may actually vary with depth and orientation within the soil. Thus, test results, which are performed near surface in a normal direction, are subject to interpretation. Plate loading tests probably yield low values in a consistent soil since density increases with depth and layering may stiffen the soil in the lateral directions. In general, increasing $k_s$ stiffened the structure, thus decreasing deflections, raising the principal frequency, and distributing pile forces more equitably.

Raising the soil modulus by an order of magnitude twice (0.01 kcl to 0.1 kcl to 1.0 kcl) resulted in significant changes in pile forces, deflections, and, especially, in frequency.

### CONCLUSIONS

A general method of analysis of three dimensional pile foundations for static or dynamic loading has been presented, suitable for computer or hand computation. Equations for the stiffness influence coefficients are given in Appendix.
1. Symmetry of piles or in pile arrangement in the group, hinged connections, or vertical piles simplify the stiffness influence coefficient equations considerably. However, since a general computer program can handle a completely general analysis as easily as a simplified problem, it is not necessary for the designer to require symmetry or to make simplifying assumptions merely for the sake of analysis. Assuming hinged connections of the piling to the cap when they are physically fixed, for example, may be unduly conservative in a static analysis or yield low values of principal frequencies.

Ignoring any lateral restraint by the soil on piling may yield completely incorrect solutions, as even a minimal soil modulus stiffens the structure considerably.

Generally, fixing the pile heads in the cap and higher values of the soil modulus stiffens the foundation structure, thus decreasing deflections, increasing the principal frequency, and apportioning loads more equitably to the piling. However, added stress resultants of shear and moment must be considered in pile design.

Results of analysis should be checked for equilibrium if they are to be verified. It should be noted that nondimensionalization, which is usually recommended for generalization and reduction of numbers to workable orders of magnitude, has been omitted in the interest of retaining physical contact with the problem. In addition, whenever merited because of unsupported length or slenderness, stability must be checked.

The method of analysis presented herein is believed to be useful for practical application in pile foundation design for static or dynamic loading.

ACKNOWLEDGMENTS

Appreciation is extended to Curtis W. Weiss of the Engineering Section of Swift and Company for checking the work reported herein.

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APPENDIX 1.—FORMULAS FOR STIFFNESS INFLUENCE COEFFICIENTS

Single pile value formulas are given. The influence coefficients are, finally

\[ S_{ij} = \sum_{k=1}^{n} S_{ijk} \]

(55)

The matrix \([S]\) is symmetrical, i.e., \(S_{ij} = S_{ji}\), as would be expected from the reciprocal law. The general equations in which the functions \(B_i\) are defined are

\[ B_1 = b_1 \cos^2 \gamma - b_2 + b_3 \sin^2 \gamma \]
\[ B_2 = 0.5 (b_{11} - b_{33}) \sin 2\gamma \]
\[ B_3 = 0.5 (b_{13} + b_{31}) \cos \gamma \sin 2\alpha \]
\[ B_0 = u_1 \sin \alpha - u_2 \cos \alpha \]
\[ B_1 = b_{11} \sin^2 \gamma + b_{22} \cos^2 \gamma \]
\[ B_2 = b_{44} \cos^2 \gamma - b_{14} + b_{44} \sin^2 \gamma \]
\[ S_{14} = B_1 \cos \alpha + b_{22} \]
\[ S_{15} = 0.5 B_1 \sin 2\alpha \]
\[ S_{16} = -B_1 \cos \alpha \]
\[ S_{17} = -u_2 B_2 \cos \alpha - B_2 \]
\[ S_{18} = u_1 B_4 \cos \alpha + (b_{14} \cos^2 \alpha - b_{44} \sin^2 \alpha) \cos \gamma \]
\[ S_{19} = B_1 B_4 \cos \alpha - u_2 b_{22} + b_{44} \sin \gamma \sin \alpha \]
\[ S_{20} = B_1 \sin^2 \alpha + b_{22} \]
\[ S_{21} = -B_2 \sin \alpha \]
\[ S_{22} = -u_2 B_3 \sin \alpha - (b_{14} \sin^2 \alpha - b_{44} \cos^2 \alpha) \cos \gamma \]
\[ S_{23} = u_1 B_3 \sin \alpha + B_3 \]
\[ S_{24} = B_1 B_4 \sin \alpha + u_1 b_{22} - b_{44} \sin \gamma \cos \alpha \]
\[ S_{25} = B_4 \]
\[ S_{26} = u_1 B_4 + b_{14} \sin \gamma \sin \alpha \]
\[ S_{27} = -u_2 B_4 + b_{14} \sin \gamma \cos \alpha \]
\[ S_{28} = B_2 B_4 \]
\[ S_{29} = u_2^2 B_4 + 2u_1 b_{14} \sin \gamma \sin \alpha \]
\[ S_{30} = B_2 B_4 - b_{14} (u_1 \sin \alpha + u_2 \cos \alpha) \sin \gamma + 0.5 B_4 \sin 2\alpha \]
\[ S_{31} = -u_2 B_2 B_4 + u_2 B_3 - u_3 \left( b_{14} \sin^2 \alpha - b_{44} \cos^2 \alpha \right) \cos \gamma \]
\[ -0.5 \left( b_{44} - b_{44} \right) \sin 2\gamma \cos \alpha \]
\[ S_{32} = u_1 B_3 B_4 + u_1 B_2 - u_3 \left( b_{14} \cos^2 \alpha - b_{44} \sin^2 \alpha \right) \cos \gamma \]
\[ -0.5 \left( b_{44} - b_{44} \right) \sin 2\gamma \sin \alpha \]
\[ S_{33} = B_1 B_4 + (u_1^2 + u_2^2) b_{14} - 2b_{44} \left( u_1 \cos \alpha + u_2 \sin \alpha \right) \sin \gamma \]
\[ + b_{44} \sin^2 \gamma + b_{44} \cos^2 \gamma \]

When a pile is symmetrical, \( I_x = I_y \), the expressions are simplified. Noting that
\[ b_1 = b_{11} = b_{12} \]
\[ b_3 = b_{14} = -b_{44} \]
\[ b_4 = b_{22} \]
\[ b_6 = b_{44} \]
the equations become
\[ B_1 = b_1 (\cos^2 \gamma - 1) + b_3 \sin^2 \gamma \]
\[ B_2 = 0.5 (b_1 - b_3) \sin 2 \gamma \]
\[ B_3 = 0 \]
\[ B_4 = u_1 \sin \alpha - u_1 \cos \alpha \]
\[ B_5 = b_1 \sin^2 \gamma + b_4 \cos^2 \gamma \]
\[ B_6 = b_1 (\cos^2 \gamma - 1) + b_6 \sin^2 \gamma \]
\[ S_{11} = B_1 \cos \alpha + b_1 \]
\[ S_{12} = 0.5 B_1 \sin 2 \alpha \]
\[ S_{13} = -B_2 \cos \alpha \]
\[ S_{14} = -u_2 B_3 \cos \alpha \]
\[ S_{15} = u_2 B_3 \cos \alpha + b_3 \cos \gamma \]
\[ S_{16} = B_1 B_4 \cos \alpha - u_2 b_3 - b_3 \sin \gamma \sin \alpha \]
\[ S_{22} = B_1 \sin^2 \alpha + b_1 \]
\[ S_{23} = -B_3 \sin \alpha \]
\[ S_{24} = -u_3 B_5 \sin \alpha - b_4 \cos \gamma \]
\[ S_{25} = u_1 B_3 \sin \alpha \]
\[ S_{26} = B_1 B_5 \sin \alpha + u_1 b_3 + b_3 \sin \gamma \cos \alpha \]
\[ S_{31} = B_2 \sin \gamma \sin \alpha \cos \alpha \]
\[ S_{33} = u_3 B_5 + b_3 \sin \gamma \cos \gamma \sin \alpha \]
\[ S_{36} = -B_3 B_4 \]
\[ S_{44} = u_2 B_3 + 2 u_4 b_3 \sin \gamma \sin \alpha + B_7 \cos \alpha + b_4 \]
\[ S_{45} = -u_1 u_3 B_4 - b_5 (u_1 \sin \alpha + u_2 \cos \alpha) \sin \gamma + 0.5 B_8 \sin 2 \alpha \]
\[ S_{46} = -u_2 B_5 \sin \gamma \cos \gamma - 0.5 (b_4 - b_3) \sin 2 \gamma \cos \alpha \]
\[ S_{54} = u_3 B_3 + 2 u_4 b_3 \sin \gamma \cos \alpha + B_7 \sin^2 \alpha + b_4 \]
\[ S_{55} = u_1 B_2 B_4 - u_2 b_3 \cos \gamma - 0.5 (b_4 - b_3) \sin 2 \gamma \sin \alpha \]
\[ S_{56} = B_1 B_2 \sin \gamma + (u_1^2 + u_2^2) b_3 + 2 b_3 (u_1 \cos \alpha + u_2 \sin \alpha) \sin \gamma \]
\[ + b_4 \sin^2 \gamma + b_6 \cos^2 \gamma \]

Hinged piling and vertical piling are common enough to be of interest. These cases are presented below.

<table>
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<th>Value</th>
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<th>Vertical Piling</th>
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<td></td>
<td>\cos \alpha = \cos \gamma = 1, \sin \alpha = \sin \gamma = 0</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>( b_{11} \cos^2 \gamma - b_{22} + b_{33} \sin^2 \gamma )</td>
<td>( b_{11} - b_{22} )</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>0.5 (( b_{11} - b_{22} )) \sin 2 \gamma</td>
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</tr>
<tr>
<td>( B_3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Value</td>
<td>Hinged Piling</td>
<td>Vertical Piling</td>
</tr>
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<td>---------------</td>
<td>----------------</td>
</tr>
<tr>
<td>$B_{1}$</td>
<td>$u_{1} \sin \alpha - u_{4} \cos \alpha$</td>
<td>$-u_{3}$</td>
</tr>
<tr>
<td>$B_{2}$</td>
<td>$b_{11} \sin^{2} \gamma + b_{22} \cos^{2} \gamma$</td>
<td>$b_{33}$</td>
</tr>
<tr>
<td>$B_{3}$</td>
<td>0</td>
<td>$b_{44} - b_{44}$</td>
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</tr>
<tr>
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</tr>
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<tr>
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<tr>
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<td>$B_{3}$</td>
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<td>$-u_{4} b_{33}$</td>
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<td>$-B_{3} B_{4}$</td>
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</tr>
<tr>
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<td>$u_{1}^{2} b_{33} + b_{44}$</td>
</tr>
<tr>
<td>$S_{42}$</td>
<td>$-u_{4} u_{2} B_{3}$</td>
<td>$-u_{4} u_{2} b_{33}$</td>
</tr>
<tr>
<td>$S_{43}$</td>
<td>$-u_{2} B_{3} B_{4}$</td>
<td>$u_{2} b_{24}$</td>
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<tr>
<td>$S_{44}$</td>
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<td>$u_{1} B_{3} B_{4}$</td>
<td>$u_{1} b_{33} - b_{33}$</td>
</tr>
<tr>
<td>$S_{46}$</td>
<td>$B_{1} B_{2} (u_{1} + u_{2}) b_{22}$</td>
<td>$u_{1}^{2} b_{11} + u_{1}^{2} b_{22} + b_{33}$</td>
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</table>

Further special cases of considerable use might be given, but are available elsewhere(1). It should be noted that although nondimensionalization is usually most desirable, it has been omitted herein for the sake of retaining a physical feeling for the problem in the hands of the designer. If nondimensionalization is desired, it can be easily carried through the foregoing expressions as shown elsewhere(1,15).

APPENDIX II.—COEFFICIENTS FOR DETERMINING PILE FORCES

The elements of matrix $[P]$, see Eq. 44, for finding pile forces are given below. The elements of $[P]$ are not symmetrical.
\[ P_{11} = b_{11} \cos \gamma \cos \alpha \]
\[ P_{12} = b_{11} \cos \gamma \sin \alpha \]
\[ P_{13} = -b_{11} \sin \gamma \]
\[ P_{14} = -b_{11} u_2 \sin \gamma - b_{13} \sin \alpha \]
\[ P_{15} = b_{11} u_3 \sin \gamma + b_{13} \cos \alpha \]
\[ P_{16} = b_{11} (u_1 \sin \alpha - u_2 \cos \alpha) \cos \gamma \]
\[ P_{17} = -b_{23} \sin \alpha \]
\[ P_{18} = b_{22} \cos \alpha \]
\[ P_{19} = 0 \]
\[ P_{20} = b_{24} \cos \gamma \cos \alpha \]
\[ P_{21} = b_{24} \cos \gamma \sin \alpha \]
\[ P_{22} = b_{25} (u_1 \cos \alpha + u_2 \sin \alpha) - b_{24} \sin \gamma \]
\[ P_{23} = b_{26} \sin \gamma \cos \alpha \]
\[ P_{24} = b_{26} \sin \gamma \sin \alpha \]
\[ P_{25} = b_{26} \cos \gamma \]
\[ P_{26} = b_{26} u_2 \cos \gamma \]
\[ P_{27} = -b_{26} u_1 \cos \gamma \]
\[ P_{28} = b_{26} (u_1 \sin \alpha - u_2 \cos \alpha) \sin \gamma \]
\[ P_{29} = -b_{26} \sin \alpha \]
\[ P_{30} = b_{24} \cos \alpha \]
\[ P_{31} = 0 \]
\[ P_{32} = b_{34} \cos \gamma \cos \alpha \]
\[ P_{33} = b_{34} \cos \gamma \sin \alpha \]
\[ P_{34} = b_{35} (u_1 \cos \alpha + u_2 \sin \alpha) - b_{34} \sin \gamma \]
\[ P_{35} = b_{36} \cos \gamma \cos \alpha \]
\[ P_{36} = b_{36} \cos \gamma \sin \alpha \]
\[ P_{37} = -b_{36} \sin \gamma \]
\[ P_{38} = -b_{36} u_2 \sin \gamma - b_{38} \sin \alpha \]
\[ P_{39} = b_{38} u_1 \sin \gamma + b_{38} \cos \alpha \]
\[ P_{40} = b_{38} (u_1 \sin \alpha - u_2 \cos \alpha) \cos \gamma \]
\[ P_{41} = 0 \]
\[ P_{42} = 0 \]
\[ P_{43} = 0 \]
\[ P_{44} = b_{46} \sin \gamma \cos \alpha \]
\[ P_{as} = b_{as} \sin \gamma \sin \alpha \]
\[ P_{as} = b_{as} \cos \gamma \]

APPENDIX III.—REFERENCES

20. Prakash, S., "Behavior of Pile Groups Subjected to Lateral Loads." Thesis presented to the
ST 5 FOUNDATIONS ANALYSIS

University of Illinois, at Urbana, Ill., in 1962, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.


APPENDIX IV.—NOTATION

The following symbols are used in this paper:

\( A \) = cross-sectional area of pile;
\( \{ A \} \) = nth principal mode;
\( [a], [a'] \) = transformation matrices;
\( E_i \) = functions defined in Appendix I;
\( [b] \) = influence coefficients matrix for a single piling called the elastic pile constant matrix herein;
\( [c] \) = statics matrix;
\( D \) = projected width of pile in direction of bending;
\( E \) = modulus of elasticity;
\( \{ F \}, \{ P \} \) = pile forces at pile;
\( G \) = modulus of elasticity in torsion;
\( h \) = batter slope of pile;
\( I_x, I_y \) = principal moments of inertia;
\( [I] \) = unity matrix;
\( [I'] \) = dynamic coupling matrix;
\( J \) = polar moment of inertia;
\( k_L \) = axial participation factor;
\( k_T \) = participation and shape factor in torsion;
\( k_s \) = subgrade modulus;
\( L \) = pile length;
\( l \) = length, variously defined within;
\( m \) = mass of pile cap and fixed appurtenances;
\( [P] = [\{ c \} [a] \} \) = pile force matrix;
\( \{ Q \} \) = pile foundation load;
\{q\} = equilibrating force at origin for single pile;
\tau = radius of gyration;
\{S\} = stiffness influence coefficient matrix for pile foundation;
\{S\}' = contribution of one pile to \{S\};
\{\tilde{S}\} = altered stiffness influence coefficient matrix used in characteristic value problem;
t = time;
t_0 = function defined by Eq. 25;
\{U\} = pile foundation coordinate axes;
\{u\}, \{u\}' = coordinates of pile center in \{U\}_1-\{U\}_2 plane;
\{x\}, \{x\}' = displacement of pile head;
\alpha = clockwise angle in horizontal plane from \{U\}_1 axis to locate direction of pile;
\beta = function defined by Eq. 11;
\gamma = direction angle of pile with vertical;
\{\Delta\} = pile foundation displacement;
\delta = 0 for hinged connection, 1 for fixed;
\kappa = function defined by Eq. 12; and
\lambda_n = characteristic values.
APPLICATIONS FOR NEW RESEARCH
FOR PILE SUPPORTED MACHINE FOUNDATIONS

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APPLICATIONS FOR NEW RESEARCH
FOR PILE SUPPORTED MACHINE FOUNDATIONS

By William E. Saul* and Thomas W. Wolf**

SYMPOSIUM

The use of piling for machine foundations can add flexibility for the
designer, help solve special problems, and possibly reduce costs. A very
complete method of analysis is presented with great flexibility in options
available as well as a catalogue of very accurate pile models. A design
for a power plant using the method is related as an example.

Keywords: Pile Foundation; Machine Foundation; Pile Analysis; Laterally
Loaded Piling; Dynamic Loads, Vibrations.

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INTRODUCTION

Foundation Characteristics

Many machine foundations are a concrete block of massive proportions and may be assumed rigid for analysis and design. It follows that the foundation may be accurately modeled as a discrete mass with six degrees of freedom, three translational and three rotational, with respect to orthogonal coordinates, usually aligned parallel to some convenient axis of the foundation. The parameters of the dynamic system are thus inertial or mass, damping, stiffness and forcing function as determined by the type of machine and the nature of its operation.

Although mechanical adjustments, operating procedures, period of operation, or some other machine parameter may be altered to improve characteristics of the foundation-machine assemblage for better performance, the properties of the operating machine are usually accepted as specified parameters of the design.

The parameters of energy dissipation mechanisms in the foundation, usually modeled as viscous damping, may be estimated or evaluated for the system for analysis. For some types of analyses an estimate of damping is required, for others damping may be neglected. In addition to the usual mechanisms of damping in structures, the dominant contribution is from soil-structure interaction. Although energy dissipators may be installed, damping is not usually considered as a design variable but rather as a factor in the analysis resulting from other design variables.

Mass and geometry determine the inertial terms in the dynamics equations. An initial layout of the machine, with consideration of a dynamic load factor or addition to the static force for estimation of
loads under operating conditions, allows calculation of bearing pressure for preliminary computations. These pressures are usually assumed uncoupled and separately estimated for vertical and rocking components, sliding and torsional motions would be considered at a later stage in the design. Thus, allowable soil bearing pressure would determine the geometry, and to some extent, the mass. Refinement of the computations may require adjustment in geometry but the layout would, in the main, be considered fixed. The foundation may be varied in thickness and pockets may be left within the block to adjust for mass and mass moment of inertia. This may be considered the prime design variable, that is, the most well understood and easily varied parameter in the system. This is especially true since the foundation may later be "tuned" or its frequency adjusted by adding mass by pouring concrete into the pockets.

If the foundation has no piling, the ratio of load to deflection in each of the six degrees of freedom may be computed from soil and bearing area data. There are a variety of suggestions or theories for this computation in addition to experimental or field measurements. These "spring constants" or uncoupled stiffness coefficients, are usually the best approximation to a linear stiffness. The models used in the computation vary from simplistic to extremely sophisticated, with the best for use in design, given due consideration for the variability and usual lack of sufficient data or low confidence level in information, being somewhere in between. As research and field data increase and improve, however, more sophisticated models will become more practical. The foundation stiffness, similar to damping, is evaluated for analysis but not usually considered as a design variable.
The machine foundation designer then may adjust damping or stiffness with limits, but mainly works with geometry and mass, which are coupled, within the constraints of the soil base on site or as altered or filled. Computations are further complicated by mass coupling or virtual mass. This is some portion of the soil which acts in concert with the concrete block as an apparent added mass. Estimation of this variable places mass in the same category as damping and stiffness as values requiring a "best estimate" for design. These methods have been well outlined by Richart\textsuperscript{1}.

**Pile Foundations**

Introduction of piling in the foundation system will affect damping, mass and stiffness\textsuperscript{2}. In addition, stiffness becomes a quantifiable design variable. The net result is that the designer has better control over at least one more variable, stiffness, perhaps a more accurate but certainly different model for computation of added mass, and may often use a less massive foundation. Reinforced concrete piling are quite competent for axial loads and in flexure to take lateral components and are most resistent to the many deleterious elements in the air-soil-water interactive zone. In addition, concrete piles can be cast to any cross-section and length desired; precast or poured in place; prismatic, stepped or tapered; reinforced or prestressed; spliced or cut; and designed as drilled caissons or utilizing standard production model driven piles. The reinforced concrete cap forms a rigid joint with the pile and since the bearing capacity of the single pile determines load capacity, the size of the foundation or cap is controlled by type of pile, pile diameter, pile spacing, and number and pattern of piles. The design of the cap itself varies only slightly, the reactive forces being concentrated at the pile locations rather than
distributed. The added mass of soil may be approximated as some percentage of that enclosed by the pile pattern above the first inflection point of the piles, probably between 30% and less than 50%. Estimation of damping is no more concise than the previous case and is primarily due to energy dissipated in pile-soil interaction. Although the foundation may be cast in contact with soil, unless the designer has reason to differ, it is assumed that resistance to lateral displacement, sliding or torsion, will be provided solely by piles. Thus frictional contact is deemed negligible. In addition, the pile cap may be cast on poor bearing soils or elevated as a platform. Thus the method of analysis presented has a wide range of applicability, from competent terrain to offshore sites, and for any type of machinery.

**THEORY**

The undamped free vibrations are expressed by the equations of motion in 6 kinematic degrees of freedom

\[ [M][\ddot{\Delta}] + [S][\Delta] = \{0\} \]  \hspace{1cm} (1)

where the 6 \( \Delta_i \) are linear displacements corresponding to the coordinates shown in Fig. 1, 3 translational and 3 rotational small amplitudes, \( \dot{\Delta}_i \) indicates differentiation with respect to time and are the acceleration components, the components of the diagonal mass matrix are mass and mass moment of inertia corresponding to translational and rotational degrees of freedom, and \([S]\) is the stiffness matrix. The concern here is primarily with the stiffness matrix. Once Eq. 1 is formed the six frequencies \( \omega_i \) and mode shapes \( \{\phi_i\} \) may be determined with the quite useful and valid assumption that damping may be taken in a form where it is a function of
If damping is desired to be included in computations, a percentage of critical damping in a viscous model in each mode may then be assumed, assessed, or measured and in this form is much more useful and understandable. In the absence of better data, damping would probably be assumed equal in each mode and modestly estimated at about 10% for fully embedded piles, less for platforms. If the circular frequency in mode \( i \) is designated \( \omega_i \) and the percentage of critical as a decimal \( \xi_i \) the damping coefficient in mode \( i \) is \( 2 \xi_i \omega_i \); this may later be converted to the damping force in mode \( i \) or viscous damping coefficients to be added as a damping force to Eq. 1, see Reference 4 for a computational method for damping as well as a method of modal numerical integration.

**Stiffness**

The stiffness matrix is a property of the structure. A coefficient \( S_{ij} \) may be defined as the force at \( i \) due to a unit displacement at \( j \) with all other displacements being zero. The subscripts \( i \) and \( j \) refer to the kinematic degrees of freedom, which coincide with the coordinate system in Fig. 1. Thus, \( S_{15} \) is a force in the direction of axis \( U_1 \) due to a unit rotation \( \Delta_5 = 1 \), which is about the \( U_2 \) axis. Further, the coefficients \( S_{15} \) in column 5 of the stiffness matrix are the complete set of forces caused by the \( \Delta_5 = 1 \) with all other \( \Delta_i = 0 \). The units of the \( S_{ij} \) are mixed, force per unit translation (kN/mm or lb/in.), force per unit rotation (kN/rad), moment per unit translation (kN-mm/mm) or moment per unit rotation (kN-mm/rad). The inverse of the stiffness matrix is a flexibility matrix \([D] = [S]^{-1}\) where its coefficients \( d_{ij} \) are the displacements at \( i \) due to a unit force at \( j \) with all other forces being zero. These
coefficients may be determined more easily by testing; thus if a force were applied to a pile foundation in one of the coordinate directions, say along \( \Delta_1 \), the 6 components of displacement may be measured and divided by the magnitude of the force applied to form the first column or vector \( \{d\}_1 \) of the flexibility matrix. If any force or load component were graphed versus a convenient component of displacement for changing increments of load during the test, the resulting curve would probably not be linear or elastic, thus the slope \( d_{ij} \) would not appear to be a constant quantity. Since the analysis is most easily and reasonably performed with the assumption of a linear elastic system, that is the coefficients \( S_{ij} \) or \( d_{ij} \) being constants, it is necessary to make assumptions to simplify the system. For the coefficients obtained by test, for example, it would be possible to cycle the load in the neighborhood of magnitude eventually expected and to use the resulting graph to obtain a secant modulus.

The stiffness of the foundation is a function of all components; soil properties, rigidity of the cap, and properties of each pile, and their configuration as a group. The foundation stiffness is, in fact, the sum of the stiffnesses of all its components and since the cap is assumed to be rigid, it is the sum of the stiffness contributions of each pile interacting with the soil. The stiffness of each pile then, in directions parallel to the kinematic degrees of freedom of the foundation, is needed. Thus

\[
[S] = \sum_{k=1}^{n} [S']_k = [S']_1 + [S']_2 + \ldots + [S']_n
\]  

(2)

where \( [S']_k \) is the stiffness of pile \( k \) of \( n \)-piles in the foundation coordinates.
Pile Stiffness

The pile stiffness is first determined in coordinates local to the pile. Extensive use of Reference 2 is useful to this section. With origin at the centroid of area at the pile head, these are formed by the longitudinal axis and the principal axes in bending. Subsequently, the pile stiffness is transformed twice, once in rotation parallel to the axis of the foundation stiffness, and secondly to account for the pile position in the foundation with respect to a coordinate center of the foundation axes. The pile stiffness may be expressed in the relationship

\[
[r]_k = [b]_k [x]_k
\]

(3)

where the \(b_{ij}\) relate forces \(F_i\) to displacements \(x_j\). The coordinate system is a triad similar to that used in Fig. 1. The form of \([b]_k\) is a sparse matrix, the diagonal elements and \(b_{24} = b_{42}\) and \(b_{15} = b_{51}\) being the only nonzero although these too may be zero depending on boundary or end conditions. Determination of the coefficients of \([b]\) follows later.

The transformations necessary are

\[
[S']_k = [c]_k [a]_k [b]_k [a]_k^T [c]_k^T
\]

(4)

where the rotation transformation matrix \([a]_k\) and the translation transformation matrix \([c]_k\) are available. Since all 3 matrices are sparse, they have been multiplied and presented in Appendix 1 so that the coefficients \(S_{ij}\) may be determined if desired by calculator or minicomputer. Note that many terms become zero if piles are vertical, are symmetric (\(I_x = I_y\)), or the end condition pinned or free to rotate.
The components of \([b]^{-1}\) may be determined experimentally by load tests as discussed earlier in terms of the foundation stiffness. Although expensive and only true for that pile under site conditions present at the time of the test, the results are tangible and do not require assumptions or hypotheses of modeling. Modeling, however, is the basic tool of analysis and necessary for design. With sufficient confidence in a particular model various pile types and configurations may be tried with only the cost of analysis. Models may be too simple so that results are doubtful and assumptions simply not up to the state of the art. Conversely, sophistication possible in research is academic; for practical design models have to be able to utilize normal data obtained from soil borings and surveys. As a function of risk and cost, this data could range from very limited to extensive. The same arguments apply to field and laboratory soil property tests. Most machines requiring consideration of dynamics in the design of their foundations would be of sufficient importance to warrant at least a boring and some tests. The primary element utilized in the various models presented herein is a linear constant in the relationship between average lateral bearing pressure and deflection, the "beam on a spring foundation" concept. This value may be assumed constant over the length of the pile or constant over an increment of depth or layer. Otherwise, it may be assumed to increase linearly with depth. It is felt that values of this constant, the modulus of subgrade reaction, may be reasonably estimated or measured at present. In addition, progress is being made in improving correlations with test data and other types of "soil modulus" concepts to eventually make it even more reasonable to use this basis for modeling and with an ever better confidence level. Determination of soil properties, especially over a period of time, appears to be the major deterrent to a higher confidence level or the use of any greater sophistication. Field
tests correlated with soil data will certainly provide the best data eventually. Finally, strength parameters are not sensitive to variation in soil modulus but displacement parameters are; thus the designer can be confident in a force parameter on a pile with an estimated soil modulus but the corresponding deflection may be considerably off. This means that improved soil data is needed for dynamic computations.

The models presented are of 2 classes; a semi-infinite pile or a finite pile. The semi-infinite models may be used whenever either parameter $\beta L$ or $\psi L > \pi$, where

$$\beta = \frac{k_s D}{4EI} \quad \text{and} \quad \psi = \frac{nD}{EI}$$

(5)

in which $k_s$ is the modulus of subgrade reaction in units of pressure per unit displacement, i.e., lb/in.$^3$ or kN/mm$^3$, and $k_s = nL$ when the subgrade modulus increases linearly with depth; $L$ is the embedded length of the pile, $EI$ the flexural rigidity of the pile as a beam, and $D$ is the projected width of the pile. There are 2 values of the parameters $\beta$ and $\psi$ for each pile, one with respect to each principal axis. Validity for use of the semi-infinite model is because of the damped wave form of the elastic curve of the pile which shows that if the pile extends beyond the first wave deflection and all other stress resultants in flexure essentially become negligible. Formulas for coefficients for 4 cases are given in Table 1 where there is a distinction made between a semi-infinite media and a pile extending as a cantilever length $L$ above the semi-infinite media. Subscripts $i$ are adjusted to correspond to the direction or axis of bending so each entry corresponds to two different coefficients unless the pile has the same properties $I_i$ and $D_i$ with respect to both principal axis. In
Table 1  Nonzero Coefficients for Semi-Infinite Piles in Flexure

<table>
<thead>
<tr>
<th>Model</th>
<th>$b_{11} &amp; b_{22}$</th>
<th>$b_{44} &amp; b_{55}$</th>
<th>$b_{15} &amp; -b_{24}$**</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Single Layer</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Beam on a* Constant Spring Fdn.</td>
<td>$(1+\delta)\kappa \beta^2$</td>
<td>$\delta \kappa$</td>
<td>$\delta \kappa \beta$</td>
</tr>
<tr>
<td>where $\kappa = 2\beta EI_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Beam on a Linearly Increasing Spring Fdn.</td>
<td>$(0.427+0.672\delta)\lambda^2 \psi_1^2$</td>
<td>$1.5036\lambda_1$</td>
<td>$\delta \lambda_1 \psi_1$</td>
</tr>
<tr>
<td>where $\lambda_1 = EI_1 \psi_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>B. Two Layer (l is the unsupported length of cantilever)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Cantilever Adjoining Model A1</td>
<td>$3[1+\delta(1+2\beta_1 \ell)]\beta^2 \kappa t_1$</td>
<td>$\delta(3+6\beta_1 \ell+6\beta_2 \ell^2)$</td>
<td>$3\delta(1+2\beta_1 \ell+\beta_2 \ell^2)\kappa \beta t_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+2\beta_1^3 \kappa t_1$</td>
<td></td>
</tr>
<tr>
<td>where $t_1 = 1/[3+6\beta_1 \ell+6\beta_2 \ell^2+(1+\delta)2\beta_1^3 \ell^3+3\beta_2 \ell^4]$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Cantilever Adjoining Model A2</td>
<td>$3(1+3\delta)\lambda^2 \psi_1^2 \beta_1 p_1$</td>
<td>$4\delta(6.918+9.204 \psi_1 \beta_2)$</td>
<td>$6\delta(3.068+3.732 \psi_1 \beta_2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+5.058 \psi_1^2 \beta_2^2 + \psi_1 \beta_2^3) \lambda_1^2 p_1$</td>
<td>$+\psi_1^2 \beta_2^2 \lambda_1 \psi_1 p_1$</td>
</tr>
<tr>
<td>where $p_1 = 1/(18.417+6\delta(1+\ell-1)) \left[6.918(1+3\delta) \psi_1 + 9.204(1+\delta) \psi_1^2 + 1.686(3+\delta) \psi_1^2 \beta_2^2 + \psi_1 \beta_2^3 \right]$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Subscripts $i$ must be adjusted for the direction or axis of bending. For the fixed condition $\delta = 1$ and for the pinned end $\delta = 0$.

** The form of $b_{24}$ is always the same as the form for $b_{15}$ but negative. The values may not be the same however since $\psi_1 \neq \psi_2$ or $\beta_1 \neq \beta_2$ unless piles have $I_1 = I_2$ and $D_x = D_y$. 
addition, the coefficients $b_{15}$ and $b_{24}$ have the same form but $b_{24}$ is always negative and, although having the same formula, they are only equal when $\beta_1 = \beta_2$ or $\psi_1 = \psi_2$. The 2-layer case is useful to account for a weak surface situation or for an elevated foundation such as a platform. The cantilever beam model for a single layer has been omitted since it is derived using a second unnecessary level of approximation. It can be easily used if desired.

**Finite Length Pile or Layered Soil**

When $\beta L$ or $\psi L < \pi$ or the soil-pile system must be modeled in layers because of varying properties of soil strata, nonlinearities in the soil modulus, or the pile is nonprismatic, a finite length beam on a spring foundation model may be used. There may be a large number of layers, but in each segment the pile is assumed to be prismatic and the soil to have a constant modulus of subgrade reaction which may be zero as a lower bound. There is no restriction on the ordering of layers, thus softer strata may underlay stiffer soils. The fourth order differential equation of a prismatic beam on a spring foundation may be solved with 8 undetermined coefficients whose values are determined by stress resultants at the joints. This results in an 8 by 8 member stiffness matrix $[K]^k$ for segment $k$ as shown in Appendix 2, where the local kinematic degrees of freedom are as shown in Fig. 2.

To obtain the stiffness matrix $[b]^k$ as used in Eq. 4 the $N$ by $N$ stiffness matrix for the pile of $M$ finite segments, as shown in Fig. 3, is first formed by adding the stiffness of each segment $[K]^k$ adjusting the kinematic degree of freedom numbers to coincide with the global numbers of Fig. 3. The resulting $N$ by $N$ stiffness matrix $[K]$ is partitioned to isolate the 4 by 4 matrix coinciding with kinematic degree of freedom
numbers 1 through 4, i.e., the surface or top end degrees of freedom, and then condensed to obtain a 4 by 4 matrix \([b_f^k]\) which coincides with the flexural degrees of freedom in \([b]^k\). The \([b]^k\) matrix is then formed by adding rows and columns with the axial and torsional coefficients, which are assumed to be uncoupled. Including axial and torsional degrees of freedom in the member of Fig. 2 and the pile of Fig. 3 could be easily done but has been omitted since it would increase the member degrees of freedom to 12 with a corresponding increase in the global degrees of freedom of 50% and, in addition, these actions are uncoupled and can be superimposed. Condensation can be accomplished by Gaussian Elimination, which is preferred in computer programming or by matrix condensation where the partitioned \(N\) by \(N\) stiffness matrix is written

\[
\begin{bmatrix}
b_{ff} & b_{fg} \\
b_{gf} & b_{gg}
\end{bmatrix}
\begin{bmatrix}
x_f \\
x_g
\end{bmatrix}
= 
\begin{bmatrix}
F_f \\
0
\end{bmatrix}
\]

(6)

and therefore

\[
[b_f^k] = [b_{ff} - b_{fg} b^{-1} b_{gf}]^k
\]

(7)

Utilizing any of the several models or from test data the pile stiffness \([b]^k\) is obtained and through use of Eqs. 4 and 2 the stiffness \([S]\) of the foundation.

**ANALYSIS**

Design of a pile foundation consists of iterative changes and analyses of a preliminary arrangement of piles. Pile type, length, number, spacing,
plan angle, and batter angle are all design variables. The vertical component takes precedence because of soil bearing and magnitude of load and helps set the number, type and spacing of piling. Greater spacing is preferred since there is less superposition of soil stresses at depth and potential settlement. Improvement in lateral stiffness may frequently be an objective and is primarily influenced by batter angle γ although increasing the flexural rigidity EI and/or projected width D of the piles helps. Rigidly connecting the pile to the foundation also results in a marked increase in stiffness when compared with a hinged connection. Thus iterative analyses with considered improvements is the design methodology. This can best be accomplished through use of a computer program.*

EXAMPLE

A foundation design was required for the Municipal Power Plant at Larned, Kansas, located in the floodplain of the Arkansas River. The decision was made to use piling as the best solution to a problem brought about by a poorly compacted deep silt soil condition and a fluctuating water table. The installation provides for flooding. Data concerning the installation are given in Table 2 and Figs. 4 and 5. Use of piling improved bearing capacity and provided lateral stiffness.

The soil conditions at the power plant site consisted of a silty clay to a depth of about 10 feet underlain by fine sand. Insufficient information was provided to determine the modulus of subgrade reaction and so estimates were made. Although several soil-pile interaction models were

---

* A program of about 600 FORTRAN statements was written for complete analysis of a general pile foundation allowing a choice of 5 pile models allowing computation of stiffness and for any kind of loading. Forces and deflections of individual piling are obtained.
Table 2. Foundation Data for Example Problem

Manufacturer  Colt Industries, Fairbanks Morse Engine Division, Beloit, Wisconsin.

Model & Engine Data  Colt-Pielstick PC-2 18 cylinder V dual fuel diesel 9000 hp. operating at 514 rpm. See photograph, Fig. 4.

Dimensions and Weight (see Fig. 5)

<table>
<thead>
<tr>
<th>Item</th>
<th>Description</th>
<th>Weight</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Engine</td>
<td>192.7 k</td>
<td>320&quot;</td>
<td>80&quot;</td>
<td>80&quot;</td>
</tr>
<tr>
<td>B</td>
<td>Alternator</td>
<td>77.0</td>
<td>50</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>Exciter</td>
<td>5.7</td>
<td>20</td>
<td>120</td>
<td>30</td>
</tr>
<tr>
<td>D</td>
<td>Foundation</td>
<td>704.5</td>
<td>544</td>
<td>132</td>
<td>129</td>
</tr>
<tr>
<td>E</td>
<td>In phase soil</td>
<td>274.3</td>
<td>544</td>
<td>132</td>
<td>60</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mass</th>
<th>Engine Unit &amp; Foundation</th>
<th>Engine Unit, Fdn. &amp; 5' of soil</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass</td>
<td>30.5 k-sec^2/ft</td>
<td>39.0</td>
</tr>
<tr>
<td>I_x</td>
<td>1230 k-ft-sec^2</td>
<td>2100</td>
</tr>
<tr>
<td>I_y</td>
<td>5250</td>
<td>7560</td>
</tr>
<tr>
<td>I_z</td>
<td>4600</td>
<td>6220</td>
</tr>
</tbody>
</table>

Soil  Poorly consolidated; silty clay to about 10 ft underlain by a fine sand with a fluctuating water level. Soil data poor and it was therefore assumed that $k_s(kci) = \sqrt{0.001z}$ where $z$ is in ft.
Table 2 (cont.)

**Computed Frequencies (radians/sec)**

a) Engine Unit & Foundation

<table>
<thead>
<tr>
<th>pile batter</th>
<th>$\omega_x$</th>
<th>$\omega_y$</th>
<th>$\omega_{zz}$</th>
<th>$\omega_{xx}$</th>
<th>$\omega_{yy}$</th>
<th>$\omega_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>24.0</td>
<td>24.0</td>
<td>27.2</td>
<td>77.9</td>
<td>159.8</td>
<td>161.7</td>
</tr>
<tr>
<td>5</td>
<td>26.0</td>
<td>28.1</td>
<td>35.9</td>
<td>79.3</td>
<td>157.5</td>
<td>159.5</td>
</tr>
<tr>
<td>4</td>
<td>26.8</td>
<td>28.7</td>
<td>39.7</td>
<td>80.4</td>
<td>156.3</td>
<td>158.3</td>
</tr>
<tr>
<td>3</td>
<td>28.1</td>
<td>29.5</td>
<td>46.3</td>
<td>82.7</td>
<td>153.8</td>
<td>155.9</td>
</tr>
<tr>
<td>2</td>
<td>30.2</td>
<td>31.2</td>
<td>59.5</td>
<td>88.4</td>
<td>147.4</td>
<td>149.8</td>
</tr>
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</table>

b) Engine Unit, Foundation & 5' of Soil

<table>
<thead>
<tr>
<th>pile batter</th>
<th>$\omega_x$</th>
<th>$\omega_y$</th>
<th>$\omega_{zz}$</th>
<th>$\omega_{xx}$</th>
<th>$\omega_{yy}$</th>
<th>$\omega_z$</th>
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</thead>
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<tr>
<td>Vertical</td>
<td>21.3</td>
<td>21.6</td>
<td>23.5</td>
<td>64.5</td>
<td>136.6</td>
<td>143.0</td>
</tr>
<tr>
<td>5</td>
<td>23.0</td>
<td>24.7</td>
<td>31.1</td>
<td>65.9</td>
<td>134.6</td>
<td>141.0</td>
</tr>
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<td>4</td>
<td>23.7</td>
<td>25.2</td>
<td>34.3</td>
<td>67.1</td>
<td>133.6</td>
<td>139.9</td>
</tr>
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<td>3</td>
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<td>25.7</td>
<td>40.1</td>
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<td>131.4</td>
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<td>2</td>
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<td>51.5</td>
<td>75.2</td>
<td>126.0</td>
<td>132.4</td>
</tr>
</tbody>
</table>

**Pile Data**

24-12 BP 53 piles of 30 to 35 ft were driven to a computed capacity of 70 to 90 kips in 3 rows. Piles in the perimeter were battered alternately at 1 to 3 or 1 to 4. Computations herein are based on an earlier trial with 33 piles in 3 rows with all perimeter piles battered, those at the corners at 45°. The piles in the study are 10" XS φ pipe spaced at 50" in the longitudinal direction and the rows 46" apart, all dimensions center-to-center of piles.
examined the one used in this report is an assumed parabolic increase with depth from zero at the base of the foundation. Such a variation seems reasonable because it attributes a low stiffness to the relatively unconfined surface layers and an increasing stiffness in the underlying sand. Broms suggests that the dynamic modulus be taken as a fraction of the soil reaction modulus for static loading, recognizing the softening effect of repetitive loading of the soil.

Mass included the machine and foundation block with all appurtenances and pockets. Computations for frequency were made with and without added soil mass. Since the inflection point for the fixed-head pile is at roughly 3/8 depth, about a third or 60 in. of this was used to compute added mass. The resulting lower frequencies with added mass are given in Table 2. The end or boundary condition of a fixed-head is obvious with a steel pile embedded in concrete. If the pile were hinged, that is, resistance to moment negligible, the stiffness and therefore frequency would be considerably decreased, in the example by a third to over a half.

Batter, or the slope of the pile as a ratio of vertical to horizontal projection, h, where h = cotγ, directly affects stiffness and therefore frequency. In Table 2 the results of trials with 4 different batter slopes and vertical are reported. The reported frequencies correspond to modes denoted by subscripts, the single subscript being translational in the direction indicated and as shown in Fig. 5 and the rotational by double subscripts about the axes indicated. Since the system is not symmetric, the center of mass does not coincide with the geometric centroid of the pile group, or center of stiffness, there is coupling in the modes not indicated in Table 2. This was most noticeable in the rotational modes
about the horizontal axes, $\omega_{xx}$ and $\omega_{yy}$, where translational and torsional components were also present.

The 2 translational modes in the horizontal plane, $\omega_x$ and $\omega_y$, and the torsional mode, $\omega_z$, all rely on lateral stiffness. There is also a strong element of this in the rotational mode, $\omega_{xx}$, because of coupling. When all piling are vertical this stiffness is supplied solely by the flexural rigidity of the piles. As the batter angle $\gamma$ is increased the component of axial rigidity of the pile participating in the horizontal plane or adding to lateral stiffness increases, adding to the flexural contribution and therefore increasing the frequency in these modes. There is a corresponding decrease in frequency in modes, $\omega_{yy}$ and $\omega_z$, relying on pile axial stiffness.

Computations were made using the computer program referred to earlier to obtain stiffness and another program to compute frequencies and modes. Values of pile forces under operating conditions were evaluated. Due to steady state operating conditions damping was not considered since resonance was avoided and dynamic forces were slightly overestimated or on the safe side. The installation has been operating since 1976.

**CONCLUSIONS**

Utilization of piling in machine foundations is shown to provide more flexibility to the designer and quite possibly result in a more economical design for some cases. It can also provide a solution to some difficult foundation problems. The key for useful utilization is a complete analysis method which provides for a choice of practical models for simulating the lateral resistance of piles as well as their axial and torsional behavior. The formulation is very general allowing location of the pile heads at different elevations, any spacing, plane angle, or
batter of piles, or a mixture of pile types or models. Solution of Eq. 1 yields frequencies which may be sufficient data for a steady state operating machine such as the example. Numerical integration may be necessary in other cases, but the required parameters are provided.

ACKNOWLEDGEMENT

Appreciation is extended to Fairbanks Morse Engine Division of Colt Industries, Beloit, Wisconsin, for permission to cite data herein.

REFERENCES


Appendix 1

Formulas for Stiffness Influence Coefficients

Formulas are given for single piles. Stiffness coefficient $S'_{ij} = S'_{ji}$ by reciprocity and function $B_i$ are defined for convenience as follows:

\[
\begin{align*}
B_1 &= b_{11} \cos^2 \gamma - b_{22} + b_{33} \sin^2 \gamma \\
B_2 &= (b_{11} - b_{33}) \sin \gamma \cos \gamma \\
B_3 &= (b_{15} + b_{24}) \cos \gamma \sin \alpha \cos \alpha \\
B_4 &= u_1 \sin \alpha - u_2 \cos \alpha \\
B_5 &= b_{11} \sin^2 \gamma + b_{33} \cos^2 \gamma \\
B_6 &= b_{44} \cos^2 \gamma - b_{55} + b_{66} \sin^2 \gamma \\
B_7 &= u_3 (b_{22} + B_1 \cos^2 \alpha) \\
B_8 &= u_3 (b_{22} + B_1 \sin^2 \alpha) \\
B_9 &= B_1 u_3 \sin \alpha \cos \alpha \\
B_{10} &= B_1 \cos \alpha \\
B_{11} &= B_2 \cos \alpha \\
B_{12} &= B_1 \sin \alpha \\
B_{13} &= B_2 \sin \alpha \\
B_{14} &= (b_{15} \cos^2 \alpha - b_{24} \sin \alpha) \cos \alpha \\
B_{15} &= b_{15} \sin \alpha \\
B_{16} &= (b_{15} \sin^2 \alpha - b_{24} \cos^2 \alpha) \cos \gamma \\
B_{17} &= u_1 b_{22} - b_{24} \sin \gamma \cos \alpha \\
B_{18} &= \sin \alpha \cos \alpha \\
B_{19} &= (b_{44} - b_{66}) \sin \gamma \cos \gamma \\
\end{align*}
\]

Thus,

\[
\begin{align*}
S'_{11} &= B_1 \cos \alpha + b_{22} \\
S'_{12} &= B_1 B_{18}
\end{align*}
\]
\[ S'_{13} = -B_{11} \]
\[ S'_{14} = -u_2 B_{11} - B_9 - B_3 \]
\[ S'_{15} = u_1 B_{11} + B_7 + B_{14} \]
\[ S'_{16} = B_4 B_{10} - u_2 b_{22} + b_{24} \sin\gamma \sin\alpha \]
\[ S'_{22} = B_{12} \sin\alpha + b_{22} \]
\[ S'_{23} = -B_{13} \]
\[ S'_{24} = -u_2 B_{13} - B_8 - B_{16} \]
\[ S'_{25} = B_3 + B_9 + u_1 B_{13} \]
\[ S'_{26} = B_4 B_{12} + B_{17} \]
\[ S'_{33} = B_5 \]
\[ S'_{34} = u_2 B_{5} + u_3 B_{13} + B_{15} \sin\alpha \]
\[ S'_{35} = -u_1 B_{5} - u_3 B_{11} - B_{15} \cos\alpha \]
\[ S'_{36} = -B_2 B_4 \]
\[ S'_{44} = u_2^2 B_5 + 2u_2 B_{15} \sin\alpha + B_6 \cos^2\alpha + b_{55} + u_3(2u_2 B_{13} + B_8 + 2B_{16}) \]
\[ S'_{45} = -u_2 u_5 + B_5 B_{18} - B_{15}(u_1 \sin\alpha + u_2 \cos\alpha) - u_3(u_2 B_{11} + u_1 B_{13} + u_3 B_{18} + 2B_3) \]
\[ S'_{46} = u_2(B_3 - B_2 B_4) - u_1 B_{16} + u_4 \cos\alpha - u_3(B_{12} B_4 + B_{17}) \]
\[ S'_{55} = u_1^2 B_5 + B_6 \sin^2\alpha + b_{55} + 2u_1 B_{15} \cos\alpha + u_3(B_7 + 2u_2 B_{11} + 2B_{14}) \]
\[ S'_{56} = u_1(B_2 B_4 + B_3) - B_{19} \sin\alpha - u_2 B_{14} + u_3[\sin\gamma(u_1 B_1 + B_{24} \sin\alpha) - u_2(B_{10} + 2B_{13})] \]
\[ S'_{66} = B_1 B_{22} + (u_1^2 + u_2^2) b_{22} - 2b_{24}(u_1 \cos\alpha + u_2 \sin\alpha) \sin\gamma + (b_{44} - b_{66}) \sin^2\gamma + b_{66} \]

Where \( U_i(u_1, u_2, u_3) \) are the coordinates of the pile top in the foundation, \( \alpha \) is the angle to the direction of batter measured clockwise in plan from the \( U_1 \) axis and \( \gamma \) is the angle of batter from the vertical in the plane of batter.
### Appendix 2

**Stiffness Matrix for a Pile Segment**

Refer to Fig. 2

\[
[K]_k = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & T_{3y} & 0 & 0 & T_{5y} & T_{4y} & 0 & 0 & -T_{6y} \\
2 & 0 & T_{3x} & -T_{5x} & 0 & 0 & T_{4x} & T_{6x} & 0 \\
3 & 0 & -T_{5x} & T_{1x} & 0 & 0 & -T_{6x} & T_{2x} & 0 \\
4 & T_{5y} & 0 & 0 & T_{1y} & T_{6y} & 0 & 0 & -T_{2y} \\
5 & T_{4y} & 0 & 0 & T_{6y} & T_{3y} & 0 & 0 & -T_{5y} \\
6 & 0 & T_{4x} & -T_{6x} & 0 & 0 & T_{3x} & T_{5x} & 0 \\
7 & 0 & T_{6x} & T_{2x} & 0 & 0 & T_{5x} & T_{1x} & 0 \\
8 & -T_{6y} & 0 & 0 & -T_{2y} & -T_{5y} & 0 & 0 & T_{1y}
\end{bmatrix}
\]

where,

- \( T_{1i} = (C'S' - CS)\kappa q \)
- \( T_{2i} = (C'S - CS')\kappa q \)
- \( T_{3i} = 2(CS + C'S')\kappa b^2 q \)
- \( T_{4i} = 2(C'S + CS')\kappa b^2 q \)
- \( T_{5i} = (S'^2 + S^2)\kappa b q \)
- \( T_{6i} = 2SS'\kappa b q \)
- \( q = 1/(S'^2 - S^2) \)
- \( C = \cos \beta L \)
- \( S = \sin \beta L \)
- \( C' = \cosh \beta L \)
- \( S' = \sinh \beta L \)
- \( \kappa = 2BEI \)
- \( \beta^4 = k_D/(4EI) \)
Figure 1. Coordinate System $U_i$ and Kinematic Degrees of Freedom $\Delta_i$.

Figure 2. Segment $k$ of Pile. Kinematic Degrees of Freedom are Local.
Figure 3. Model of Pile of M Finite Segments
Figure 4. Photograph of Colt-Pielstick V 18 Diesel Installed for Power Plant on Pile-Supported Foundation at Larned, Kansas
NOTES ON PILE STIFFNESS

I. Axial

1. H.G. Poulos: [see "Load-Settlement Prediction for Piles and Piers", H.G. Poulos, ASCE Journal SM9, Sept. 72, pp. 879-895]. Based on theory of elasticity, formulation as a rigid body in a semi-infinite mass, extended for compressible pile, later for nonhomogeneous soil. Used to predict load-settlement curve to failure

\[ b_{33} = \frac{F_i}{Y_i} = \frac{E_s d}{I} \]

below "yield"

\( E_s \) = modulus of elasticity of soil
\( d \) = pile diameter
\( I \) = pile setttlement influence factor

\[ = I_1 R_i R_i R_i \] where each variable is obtained from a prepared curve.


\[ b_{33} = \frac{P}{\rho} = \frac{L E_s}{I_p} \]

where \( L \) is pile length, \( E_s \) the soil modulus, and \( I_p \) an influence settlement factor. See Fig. 10.6 of paper (p. 347) for chart to determine \( I_p (K, L/D) \) for a floating pile. There are other charts for corrections for end bearing, group action, etc.

2. M. Novak: [see "Dynamic Stiffness and Damping of Piles" M. Novak, Can. Geotech. J., Vol. 11, 1974, pp. 574-598]. Based on a model which assumes soil to be composed of
Notes on Pile Stiffness

independent infinitesimally thin horizontal layers that extend to infinity, a generalized beam-on-spring-foundation model. Pile may be short but is assumed hinged at tip, i.e., tip does not move vertically

\[ b_{33} = \frac{E_P A}{r_0 f_{18,1}} \]

\[ f_{18,1} = \frac{F_{18}(\Lambda)}{\pi r_0} \]

where

- \( E_P \) = modulus of elasticity of pile
- \( A \) = area of pile
- \( r_0 \) = radius of pile

\[ f_{18,1} = f_{18,1} \left( \frac{V_s}{V_c}, \frac{\pi}{r_0} \right) \] are given in Fig. 9 of paper

\[ V_s = \sqrt{\frac{G}{\rho}} = \text{shear wave velocity of soil} \]

\[ G = \text{shear modulus of elasticity of soil} \]

\[ \rho = \frac{w}{g} = \text{mass density of soil \ (e.g., lb-sec}^2) \]

\[ V_c = \sqrt{\frac{E_P}{\rho_P}} = \text{longitudinal wave velocity in pile} \]

\[ \rho_P = \text{mass density of pile} \]

\[ F_{18}(\Lambda) = \Lambda \cot \Lambda = F_{18}(\Lambda)_1 + i F_{18}(\Lambda)_2 \]

\[ A = \frac{1}{\pi \sqrt{E_P A}} \left[ u \omega^2 - G S_{\omega_1} - i(c \omega + G S_{\omega_2}) \right] \]

in which \( \omega \) = excitation frequency

\[ u = \text{mass of pile per unit length} \]

\[ c = \text{coef. of pile internal damping} \]

\( S_{\omega_1} \) \& \( S_{\omega_2} \) are Bessel functions

For static conditions \( \omega = 0 \) leads to \( \Lambda = L \frac{4\pi r G}{E_P A} \)

\[ f_{18,1} = \frac{r_0}{L} \Lambda \coth \Lambda \]
Notes on Pile Stiffness


\[ x_3 = w_s + w_{pp} + w_{ps} \]

where

\[ w_s = (Q_p + a_s Q_s) \frac{L}{A_p E_p} \]  

deformation of pile

\[ w_{pp} = \frac{q_p B}{E_s I_{pp}} \cdot \frac{C_p Q_p}{B q_o} \]  
pile point settlement due to point load

\[ w_{ps} = \frac{f_s B}{E_s I_{ps}} \cdot \frac{C_s Q_s}{D q_o} \]  
pile point settlement due to shear

\[ Q_p = \text{point load} = \beta Q_t \]

\[ Q_s = \text{shear load} = (1-\beta) Q_t \]

\[ Q_p + Q_s = Q_t = \text{axial load on pile} \]

\[ a_s = a_s (\text{diam., distribution of skin friction}) \]

\[ \frac{1}{10} < a_s < \frac{2}{3} \] (approx.) use \( a_s = 1/2 \) for uniform skin friction & prismatic pile.

\[ L, A_p, E_p \] = pile properties

\[ I_{pp} \text{ & } I_{ps} \] are influence coefs. from theory of elasticity (see Poulos).

\[ B = \text{diam. of pile} \]

\[ D = \text{embedded length of pile} \]

\[ E_s = E_s/(1-\nu_s^2) \] = plane strain modulus of elasticity of soil,

\[ \nu_s = \text{Poisson's ratio} \]

\[ q_p = \text{net pressure on pile point} \]

\[ f_s = \text{average shear on pile surface (skin friction)} \]

\[ C_p \text{ & } C_s \] are empirical coefficients.
Notes on Pile Stiffness

where \( C_p \) comes from Table 6 and

\[
C_S = (0.93 + 0.16\nu D/B) C_p
\]

\( q_o \) = ultimate point resistance.

Therefore,

\[
\frac{1}{B_{33}} = [\beta + \sigma_s (1-\beta)] \frac{L}{A_p E_p} + \frac{C_p \beta}{B q_o} + \frac{C_s (1-\beta)}{D q_o}
\]


\[
b_{33} = \frac{P_t}{w_t} = G \xi r_o \left[ \frac{4}{\eta (1-\nu)} + \frac{2\xi}{\xi} \frac{L}{r_o} \frac{\tanh(\mu L)}{\mu L} \right] + \frac{4}{\eta (1-\nu)} \frac{1}{\pi \xi} \frac{L}{r_o} \frac{\tanh(\mu L)}{\mu L}^{-1}
\]

where

\[
\xi = \ln(r_m/r_o) = \ln [2.5(\ell/r_o) \rho (1-\nu)]
\]

\[
\mu \ell = \sqrt{2/(\xi \lambda)} (\ell/r_o)
\]

\( \rho = G(\ell/2)/G(\ell) \) (ratio of \( G \) at midheight to \( G \) at tip)

\( \eta = 1 \)

\( G = \) soil modulus

\( r_o = \) pile radius

\( \nu = \) Poisson's ratio of soil

\( \lambda = E_p/G_s \)

\( \ell = \) pile length, \( E_p = \) pile modulus
Notes on Pile Stiffness

5. Examples:

Given: Pile; Solid cylinder, concrete, $E_p = 3000$ ksi, $L = 25$ ft,
$D = 1$ ft. soil; loose sand, $E_s = 3.5$ ksi, $\nu = 0.4$;
$\gamma = 110$ pcf, uniform.

(a) Poulos.

$L/d = 25 \Rightarrow \text{Fig. 1} \Rightarrow I_1 = 0.075$

$K = \frac{E_p R_A}{E_s} = \frac{3000}{3.5}(1) \quad \text{Solid Pile}, R_A = 1$

$= 860 \quad \text{Fig. 3} \Rightarrow R_k = 1.2$

$R_h = 1$ for pile in a semi-infinite mass

$R_b = 1$ for a uniform soil

$I = I_1 R_h R_b = 0.075(1.2)(1)(1) = 0.09$

Ratio of load carried by base

$\beta = \beta_1 C_k C_b = 0.057(0.92)(1) = 0.0525$

$\beta_1 \Rightarrow \text{Fig. 2}, \quad C_k \Rightarrow \text{Fig. 4}$

$b_{33} = \frac{E_s d}{I} = \frac{3.5(12)}{0.09} = 467\text{k/in.}$

or, using Fig. 10-6 of 2d paper: $b_{33} = \frac{E_s L}{I_p} \frac{3.5 \times 25 \times 12}{2.2} = 477\text{k/in.}$

(b) Novak.

$L/r_o = \frac{25}{.5} = 50 \quad \rho_s = \frac{110}{32.2} = 3.4161 \frac{\text{lb-sec}^2}{\text{ft}^4}$

$\rho_p = \frac{145}{32.2} = 4.5031 \quad "$

$G = \frac{E_s}{2(1+\nu)} = \frac{3.5}{2.8} = 1.25 \text{ ksi}$

$V_s = \sqrt{\frac{G}{\rho_s}}$

$= \frac{1.25(1000)(144)}{3.4161} = 230 \text{ ft/sec} = 2755. \text{ in/sec}$
Notes on Pile Stiffness

\[ v_c = \gamma \frac{E_p}{\rho_p} = \sqrt{\frac{3000(1000)(144)}{4.5031}} = 9795 \text{ ft/sec} \]

\[ \frac{v_s}{v_c} = \frac{230}{9795} = 0.0234 \]

\[ :. \text{ Fig. 9 } \Rightarrow f_{18,1} = 0.025 \]

\[ b_{33} = \frac{E_p A}{r_o} f_{18,1} = \frac{3000(\pi)(6)^2}{6}(0.025) = 1414 \text{ k/in} \]

or, \[ \Lambda = L \sqrt{\frac{4\pi r_o G}{E_p A}} = 25 \times 12 \sqrt{\frac{1.25(4\pi)6}{3000(\pi)6^2}} = 5, \quad \Lambda \coth \Lambda = 5.00045 \]

\[ b_{33} = \frac{E_p A F_{18}}{L/r_o} = \frac{E_p A}{L} F_{18} = \frac{5E_p A}{L} = 5654 \text{ k/in.} \]

(c) Vesic

from p 5 \( \beta = 0.0525 \); from Eq. 15 Randolph & Wroth

\[ \beta = \left[ 1 + \frac{\pi n(1-\nu) \rho}{2 \xi} \right]^{-1} \]

\[ \xi = \ln[2.5(\frac{G}{r_o})^\nu(1-\nu)] = \ln[2.5(50)(1)(.6)] = 4.32 \]

\[ \beta = \left[ 1 + \frac{\pi(1)(.6)(50)}{2(4.32)(50)} \right]^{-1} = 0.084 \]

thus, \( 0.05 < \beta < 0.08 \), will use \( \beta = 0.08 \)

\[ q_o = cN_c + q_v N_q = 0(N_c) + .11(25)(20) \quad N_{est.} @ 20 \]

\[ = 55 \text{ k/ft}^2 = .382 \text{ k/in}^2 \]

Table 6 \( \Rightarrow C_p \approx 0.04 \quad a_s \text{ for sand } = 2/3 \)

\[ C_s = (0.93 + 0.16 \sqrt{\frac{25}{1}}) 0.04 = 0.069 \]

\[ b_{33} = \left[ (\beta + a_s (1-\beta)) \frac{L}{E_p} \right] \left[ \frac{C_p \beta}{E_q} + \frac{C_s (1-\beta)}{D} \right]^{-1} \]

\[ = \left[ (.08 + \frac{2}{3}(1-.08)) \frac{2.5(12)}{\pi^2 (3000)} + \frac{0.04(0.08)}{12(.382)} + \frac{0.069(1-.08)}{25(12)(.382)} \right]^{-1} \]

\[ = 536 \text{ k/in.} \]

(d) Randolph & Wroth

\[ \xi = 4.32 \text{ (see above) } \quad \lambda = \frac{E_p}{G_o} = \frac{3000}{1.25} = 2400 \]
Notes on Pile Stiffness

\[ u_l = \sqrt{\frac{2}{\zeta \lambda}} \frac{L}{R_o} = \sqrt{\frac{2}{4.30(2400)}} (50) = .694 \]

\[ \tanh(u_l) = 0.6010 \]

\[ b_{33} = 1.25(6) \left[ \frac{4}{6} + \frac{2\pi (1)(50)(.6010)}{4.3175(.694)} \right] \left[ 1 + \frac{4}{6} \frac{1(50)(.6010)}{(2400)(.694)} \right]^{-1} = 503 \text{ k/in.} \]

(e) Summary

\[ b_{33} = k_L (\frac{AE}{L}) = \frac{3000(\pi 6^2)}{25(12)} k_L = 1131 k_L \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( b_{33} )</th>
<th>( k_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poulos</td>
<td>467-477 k/in.</td>
<td>0.41-0.42</td>
</tr>
<tr>
<td>Novak*</td>
<td>1414-5654</td>
<td>1.25-5.00</td>
</tr>
<tr>
<td>Vesic</td>
<td>536</td>
<td>0.47</td>
</tr>
<tr>
<td>Ran &amp; Wr</td>
<td>503</td>
<td>0.44</td>
</tr>
<tr>
<td>Saul(68)*</td>
<td>1131</td>
<td>1.00</td>
</tr>
</tbody>
</table>

*Both in error, assumes no pile tip displacement.

II. Torsion

1. M.W. O'Neill: [see Discussion in ASCE ST2 Feb. 1969]

\[ b_{55} = \sqrt{4\pi r^2 G_s G_p J} \]

where \( r \) = pile radius

\( G_s \) = Soil shear modulus of elasticity

\( G_p \) = Pile " " " "

\( J \) = torsional constant of pile

Derived from strength of materials approach assuming elastic pile & soil.

Notes on Pile Stiffness

\[ b_{66} = G_s d^3 \frac{F_\phi}{I_\phi} \]

where \( G_s \) = soil modulus
\( d \) = pile diameter
\( I_\phi = I_\phi (K_T, L/d) \) = influence coefficient
\( L \) = pile length

\[ K_T = \frac{G_p J_p}{G_s d^4} \]

\( J_p \) = pile torsional constant
\( G_p \) = pile shear modulus of elasticity
\( F_\phi = F_\phi \left( \frac{T}{T_u}, K_{Te} \right) \), \( K_{Te} = K_T \left( 25/L/d \right)^2 \)

\[ 0.8 < F_\phi < 1.0 \]

soil slip factor


fixed tip \[ b_{66} = \frac{G_p J}{r_o} f_{T,1} \), \( f_{T,1} = \frac{F_T (\Lambda)}{L/r_o} \) see Fig. 5

pinned tip substitute \( F_T (\Lambda) \) for \( F_T (\Lambda) \), no curve given

May be written \[ b_{66} = \frac{G_p J}{L} \omega L \coth \omega L, \omega = \sqrt{\frac{4\pi r^2 G_s}{G_p J}} \]

4. Torsion examples

See Section I part 5 of these notes for data.

\[ J = \frac{\pi d^4}{32} \quad \text{(polar moment of inertia for circle).} \]

a) O'Neill

\[ G_p = \frac{E_p}{2(1+v)} = \frac{3000}{2(1+.25)} = 1200 \text{ ksi} \]

\[ J = \frac{\pi (12)^4}{32} = 2036 \text{ in.}^4 \]
Notes on Pile Stiffness

\[ b_{66} = \sqrt{4\pi (6)^2 (1.25) (1200) (2036)} = 37,168 \text{ in-k/radian} \]

(b) Poulos

\[ K_T = \frac{G_p J_p}{G_s d^4} = \frac{1200 (2036)}{1.25 (12)^4} = 94.26, \quad L/d = 25 \]

Fig. 4 \[ I_\phi = 0.052, \quad F_\phi \text{ assumed 1 (no slip)} \]

\[ b_{66} = G_s d^3 \frac{F_\phi}{I_\phi} = 1.25 (12)^3 \frac{1}{0.052} = 41,536 \text{ in-k/radian} \]

c) Novak and Howell

\[ \frac{V_s}{V_c} = 0.0234 \text{ (Sect. I, Pt. 5b)} \]

\[ Q_o = r_o \omega \sqrt{\rho / G} \text{ dimensionless frequency} \]

\[ \omega = \sqrt{\frac{4 \pi r^2 G_s}{G_p J_p}} = \sqrt{\frac{4 \pi (6)^2 1.25}{203.6 (1200)}} + 0.0152 \]

\[ \omega L = 4.564, \quad \coth(\omega L) = 1.0002, \quad K_T = 4.565 \]

\[ b_{66} = \frac{G_o J \omega L \coth(\omega L)}{L} = \frac{1200 (2036) (4.565)}{25 \times 12} = 37,170 \text{ in-k/Rad} \]

cr, \[ \sqrt{\frac{G_s}{2G_p}} = \sqrt{\frac{1.25}{2 (1200)}} = 0.0228 \Rightarrow \text{Fig. 6 yields} \]

\[ f_{\tau, 1} = 0.0911 \quad \therefore \quad b_{66} = \frac{G_p J}{r_o f_{\tau, 1}} = \frac{1200 (2036) (0.0911)}{6} \]

\[ = 37,100 \text{ in-k/Rad} \]

d) Summary

\[ b_{66} = K_T \frac{JG}{L} = \frac{2036 (1200)}{25 (12)} \quad K_T = 8140 \text{ K}_T \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( b_{66} )</th>
<th>( K_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>O'Neill</td>
<td>37,200</td>
<td>4.57</td>
</tr>
<tr>
<td>Poulos</td>
<td>41,500</td>
<td>5.10</td>
</tr>
<tr>
<td>Novak &amp; H.</td>
<td>37,100</td>
<td>4.56</td>
</tr>
<tr>
<td>Saul</td>
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<td>1.00</td>
</tr>
</tbody>
</table>
Computerv study of dynamic behavior of piling

By Charles H. Samson, Jr., 1 M. ASCE, Teddy J. Hirsch, 2 M. ASCE, and Lee L. Lowery, Jr., 3 A. M. ASCE

Synopsis

The application of wave theory to the investigation of structural behavior of piling is examined. A digital-computer program based substantially on the work of E. A. L. Smith was used in generating the theoretical solutions. The essentials of Smith's development are given, followed by a discussion of certain extensions and applications of the procedure.

It is illustrated how, through the use of high-speed digital computation, the influence on pile behavior of factors such as ram weight, ram velocity, diesel-hammer pressure, capblock and cushion block stiffnesses, pile material properties, and soil properties may be evaluated.

Comparisons are made with the "exact" solution for an ideal bar and with experimental results from a field test. The effects of segment length and time interval for the discrete-element solution are examined. The use of automatic plotting capability of the digital computer is illustrated.

Note.—Discussion open until January 1, 1964. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 88, No. ST4, August, 1963.

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INTRODUCTION

The study of behavior of piling has received considerable attention by investigators in the past. Much work has been directed toward establishing simplified formulas, both empirical and semi-rational. It is generally recognized that none of these proves completely satisfactory for the broad spectrum of pile types, pile drivers, and soil conditions encountered in present-day foundation problems. For a discussion of pile formulas, in general, the reader is referred to the work of Robert D. Chellis.4

The purpose of the present paper is to examine recent developments in the use of wave theory in the structural analysis of piling. In the work of Smith5 a major contribution has been made toward adapting the general theory of stress-wave propagation to the peculiar problems associated with piling. One of the major objections that has been raised against this approach is the need for high-speed electronic computation in the practical application of the procedure. In the present paper, the writers endeavor to demonstrate that considerable useful data can be developed through generalized parameter studies. Certain extensions and refinements to Smith's work are also described. It is emphasized that the present study is limited to structural behavior of piling during driving. An effort has not been made to correlate resistance to penetration with ultimate bearing capacity.

D. V. Isaac's6 is thought to be the first (1931) to note the occurrence of wave action in piling during driving. However, it is interesting to note that Isaac Todhunter and Karl Pearson7 state that H. Moseley8 in 1843 gives driving of piles as a special application of impact and resilience theory.

In 1938, W. H. Glenville, G. Grime, E. N. Fox, and W. W. Davies9 published the results of extensive mathematical and experimental studies of piling in which stress-wave propagation was considered. Because of limited computational capability at that time, the application of wave theory necessarily involved simplifying assumptions. Nevertheless, this work has considerable value. In 1940 A. E. Cummings10 examined dynamic pile-driving formulas, in general, and provided a brief description of the wave-theory approach and the theoretical work of Glenville and his associates.

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3 6 "Reinforced Concrete Pile Formulas," by D. V. Isaac's, Transactions, Institution of Engineers, Australia, Vol. 12, 1931, p. 312.

THEORY

Wave Equation.—The wave-theory approach to the problem does not involve a "formula" in the usual sense. The basis for the procedure is the classical one-dimensional wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1} \]

in which \(c\) represents the velocity of propagation of strain wave along bar \((\sqrt{E/\rho}); x\) is the direction of longitudinal axis; \(u\) represents the displacement of bar cross section in \(x\) direction; \(t\) denotes time; \(E\) is the modulus of elasticity of material; and \(\rho\) is the mass per unit volume of material.

Wave theory as applied to problems of longitudinal wave transmission and impact is investigated in detail by L. H. Donnell,11 S. Timoshenko and J. N. Goodier,12 H. Kolsky,13 and H. N. Abramson, H. J. Plass, and E. A. Rippner.14 Julius Mikowitz15 provides an extensive list of references on the broad subject of elastic wave propagation.

Smith's Idealization.—Fig. 1 illustrates the idealization of the pile system suggested by Smith. In general, the system is considered to be composed of [see Fig. 1(a)] the following: 1. A ram, to which an initial velocity is imparted by the pile driver; 2. a capblock (cushioning material); 3. a pile cap; 4. a cushion block (cushioning material); 5. a pile; and 6. the supporting medium, or soil. Fig. 10(b) shows the idealizations for the various components of the actual pile. The ram, capblock, pile cap, cushion block, and pile are pictured as appropriate discrete weights and stiffnesses. The frictional soil resistance on the side of the pile is represented by a series of side springs; the

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point resistance is accounted for by a single spring at the point of the pile. The characteristics of these various components will be described in greater detail.

Actual situations may differ from that in Fig. 1. For example, a cushion block might not be used or an anvil may be placed between the ram and cap-block. Such cases are readily accommodated.

**FIG. 1.—IDEALIZATION OF A PILE FOR PURPOSE OF ANALYSIS**

*Internal Springs.*—The ram, cap-block, pile cap, and cushion block may in general be considered to consist of "internal springs," although in the representation of Fig. 1(b) the ram and the pile cap are treated as though rigid (a reasonable assumption for many practical cases).

Figs. 2(a), 2(b), 2(c), and 2(d) suggest different possibilities for representing the load-deformation characteristics of the internal springs. In Fig. 2(a) the material is considered to experience no internal damping. In Figs. 2(b) and 2(d), the material is assumed to have internal damping according to the linear relationships shown. In Fig. 2(c), loading and unloading are consid-
Eroded to occur along a hysteresis loop. The assumption that should be made for a given problem depends on the material and its known load-deformation behavior under dynamic conditions. Future investigation will probably shed more light on this subject and will indicate which of Figs. 2(a), 2(b), 2(c), and 2(d) is most realistic for a given case. Furthermore, other representations can be incorporated if desired.

**External Springs.** — The resistance to dynamic loading afforded by the soil in shear along the outer surface of the pile and in bearing at the point of the pile is not clearly understood. Future studies will provide more information on this question. Fig. 3 shows the load-deformation characteristics assumed for the soil in Smith's procedure, exclusive of damping effects. The path OABCDEF is a representation of loading and unloading in side friction. In the present paper, both side friction and point-bearing forces are permitted to act on the last pile segment. In this case, the point-bearing force must be prevented from exerting tension on the pile. Thus, loading and unloading would occur along OABCFCB. Smith follows a somewhat different approach in combining point bearing and side friction.

It is seen that the characteristics of Fig. 3 are defined essentially by the quantities "Q" and "Ru." "Q" is termed the quake and represents the maximum deformation that may occur elastically. "Ru." is the ultimate ground resistance, or the load at which a spring behaves purely plastically.

A load-deformation diagram of the sort of Fig. 3 may be established separately for each spring. Thus,

\[
K'(m) = \frac{Ru(m)}{Q(m)} \quad \text{(2)}
\]

in which \(K'(m)\) is the spring constant (during elastic deformation) for external spring \(m\).

**Basic Equations.** — Eqs. 3 through 7 were developed by Smith:

\[
D(m,t) = D(m,t - 1) + 12At \cdot V(m,t - 1) \cdot f(t) \quad \text{(3)}
\]

\[
C(m,t) = \frac{D(m,t)}{D(m,t + 1, t)} \quad \text{(4)}
\]

\[
F(m,t) = C(m,t) K(m) \quad \text{(5)}
\]

\[
V(m,t) = V(m,t - 1) + \frac{F(m,t) - R(m,t)}{W(m)} \quad \text{(6)}
\]

in which \(\cdot\) represents a functional designation; \(m\) denotes the element number; \(t\) is the time interval number; \(At\) is the size of time intervals in seconds; \(C(m,t)\) represents the compression of internal spring \(m\) in time interval \(t\), in inches; \(D(m,t)\) describes the displacement of element \(m\) in time interval \(t\), in inches; \(D(m,t)\) represents the plastic displacement of external spring \(m\) in time interval \(t\), in inches; \(F(m,t)\) is the force in internal spring \(m\) in time interval \(t\), in pounds; \(g\) represents the acceleration due to gravity, in feet per second squared; \(J(m)\) is the damping constant of soil at element \(m\), in seconds per foot; \(K(m)\) is the spring constant associated with internal spring \(m\), in pounds per inch; \(K'(m)\) is the spring constant associated with external spring \(m\), in pounds per inch; \(R(m,t)\) is the force exerted by external spring \(m\) on element \(m\) in time interval \(t\), in pounds; \(V(m,t)\) represents the velocity of element \(m\) in time interval \(t\), in feet per second; and \(W(m)\) is the weight of element \(m\), in pounds. This notation differs slightly from that used by Smith.

Also, Smith restricts the soil damping constant \(J\) and the quake \(Q\) to two values each one, for the point of the pile in bearing and one for the side of the pile in friction. Although present knowledge of the damping behavior of soils perhaps does not justify greater refinement, \(J\) and \(Q\) are treated herein as functions of \(m\) for the sake of generality.

The use of a spring constant \(K(m)\) implies a load-deformation behavior of the kind shown in Fig. 2(a). For this situation, \(K(m)\) is the slope of the straight line. Smith develops special relationships for internal damping in the capblock and the cushion block. He obtains the following equation for alternative use with Eq. 5:

\[
F(m,t) = \frac{K(m)}{[e(m)]^2} C(m,t) - \left(1 - \frac{1}{[e(m)]^2} \right) K(m) C(m,t)_{max} \quad \text{(8)}
\]

in which \(e(m)\) is the coefficient of restitution of internal spring \(m\); and \(C(m,t)_{\text{max}}\) denotes the temporary maximum value of \(C(m,t)\). With reference to Fig. 1, Eq. 8 would be applicable in the calculation of the forces in internal springs \(m = 1\) and \(m = 2\) in the ranges ABC and DE in Fig. 2(d). The load-deformation relationship characterized by Eqs. 5 and 8 is illustrated by the path OABCDEOF. For a pile up on a cushion block no tensile forces can exist; consequently, only this part of the diagram applies. Intermittent unloading-loading is typified by the path ABC, established by control of the quantity \(C(m,t)_{\text{max}}\) in Eq. 8. The slope of lines AB, BC, and DE depends on the coefficient of restitution \(e(m)\).

The computations proceed as follows:

1. The initial velocity of the ram is determined from the properties of the pile driver. Other time-dependent quantities are initialized at zero.
2. Displacements \(D(m,1)\) are calculated by Eq. 3. It is to be noted that \(V(1,0)\) is the initial velocity of the ram.
3. Compressions \(C(m,1)\) are calculated by Eq. 4.
4. Internal spring forces \(F(m,1)\) are calculated by Eq. 5 or Eq. 8 as appropriate.
5. External spring forces \(R(m,1)\) are calculated by Eq. 6.
6. Velocities \(V(m,1)\) are calculated by Eq. 7.
7. The cycle is repeated for successive time intervals.

In Eq. 6 plastic deformation \(D'(m,t)\) for a given external spring follows Fig. 3 and may be determined by special routines. For example, when \(D(m,t)\) is less than \(Q(m)\), \(D'(m,t)\) is zero; when \(D(m,t)\) is greater than \(Q(m)\) along line AB (see Fig. 3), \(D'(m,t)\) is equal to \(D(m,t) - Q(m)\).

Smith notes that Eq. 6 produces no damping when \(D(m,t) - D'(m,t)\) becomes zero. He suggests an alternate equation to be used after \(D(m,t)\) first becomes
equal to \( Q \) is:

\[
R(m,t) = \left[ \frac{D(m,t) - D'(m,t)}{V(m,t) - V'(m,t)} \right] K'(m) + J(m) K''(m) Q(m) V(m,t - 1) \quad \ldots \quad (9)
\]

Care must be used to satisfy conditions at the head and point of the pile. Consider Eq. 5. When \( m = p \), in which \( p \) is number of the last element of the pile, \( K(p) \) must be set equal to zero since there is no \( F(p,t) \) force (see Fig. 1). Also, at the point of the pile, the soil spring must be prevented from exerting tension on the pile point. In applying Eq. 7 to the ram \( (m = 1) \), \( F(0,t) \) should be set equal to zero.

For the idealization of Fig. 1, it is apparent that the spring associated with \( K(2) \) represents both the cushion block and the top element of the pile. It may be obtained by the following equation:

\[
\frac{1}{K(2)} = \frac{1}{K(2)_{cushion}} + \frac{1}{K(2)_{pile}} \quad \ldots \quad (10)
\]

A more complete discussion of digital-computer programming details and recommended values for various physical quantities are given by Smith. It may be noted that the wave equation (Eq. 1) does not appear explicitly in the basic equations given for Smith's idealization. However, Eqs. 3-7 may be combined to produce a difference equation corresponding to Eq. 1 with resistance included.

**APPLICATIONS AND EXTENSIONS OF THEORY**

**General.**—The basic equations previously presented have been programmed for use as an analytical tool both in practical engineering applications and also in research. As a means of describing certain extensions to the procedure as well as illustrating its use in investigating the influence of various parameters, several situations have been studied. Fig. 4 identifies four different piles: piles I, III, and IV are prestressed concrete; pile II is a steel H-section. For each pile, different cases of soil resistance are considered.

**Application to Ideal Pile;PILE STUDIES.**—For purposes of comparing Smith's procedure with the exact method of analyzing elastic bars, Pile I is considered. Two cases are treated: in the first, the point is free; in the second, the point is fixed. It is noteworthy that exact solutions are feasible only for pile problems involving special conditions of material properties and soil resistance. However, problems encountered in practice do not usually fall into this category.

The following information, in addition to that given in Fig. 4, is applicable to the Pile I studies: \( W(ram) = 11,500 \text{ lb}; \) \( V(ram,0) = 14.45 \text{ ft per sec}; \) \( K(\text{cushion block}) = 3,930,000 \text{ lb per in.}; \) and \( e(\text{cushion block}) = 1.00. \)

Figs. 5 and 6 show stress versus time plots at the mid-length of the pile for free and fixed end conditions, respectively. Solutions have been obtained for the exact solution of the one-dimensional wave equation (Eq. 1) and for Smith's discrete-element method using ten, twenty, and forty pile segments. The term "exact" as used here refers to the direct solution of Eq. 1. The

**FIG. 4.—PILES AND ULTIMATE SOIL RESISTANCES USED IN STUDIES**

\[\text{NOTE:} \quad \text{NUMERICAL VALUES INDICATE TOTAL POINT AND SIDE RESISTANCES IN KIPS, DISTRIBUTED AS ShOWN.}\]
value of $\Delta t$ used for all Smith's solutions is $1/10,000$ sec. It is interesting to note how increasing the number of pile segments improves the accuracy of the discrete-element solution, as might be expected.

The accuracy of the discrete-element solution is also related to the size of the time increment $\Delta t$. W. P. Heising, in his discussion of the equation of motion for free longitudinal vibrations in a continuous elastic bar, points out that the discrete-element solution is an exact solution of the partial differential equation when

$$\Delta t = \frac{\Delta L}{E} \sqrt{\frac{E}{\rho}} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (11)$$

in which $\Delta L$ is the segment length. Smith draws a similar conclusion. Figs. 7 and 8 show comparisons of maximum tensile stress and maximum compressive stress, respectively, versus position along the length of the pile for various increments of time. These solutions use a segment length of 9 ft. If a time increment larger than that given by Eq. 11 is used, the discrete-element solution will diverge and no valid results can be obtained. As noted by Smith, in this case the numerical calculation of the discrete-element stress wave does not progress as rapidly as the actual stress wave. Consequently, the value of $\Delta t$ given by Eq. 11 is called the "critical" value.

Heising has also noted that when $\Delta t < \Delta L/\sqrt{E/\rho}$ is used in a discrete-element solution, a less accurate solution is obtained for the continuous bar. As $\Delta t$ becomes progressively smaller, the solution approaches the actual behavior of the discrete-element system (segment lengths equal to $\Delta L$) used to simulate the pile.

This, in general, leads to a less accurate solution for the longitudinal vibrations of a slender continuous bar. If, however, the discrete-element system were divided into a large number of segments, the behavior of this simulated pile would be essentially the same as that of the slender continuous bar irrespective of how small $\Delta t$ becomes provided $\Delta L/\sqrt{E/\rho} \gg \Delta t > 0$. This means that if the pile is divided into only a few segments, the accuracy of the solution will be more sensitive to the choice of $\Delta t$ than if it is divided into many segments. For practical problems, a choice of $\Delta t$ equal to approximately one-half the "critical" value appears suitable because inelastic springs and materials of different densities and elastic moduli are usually involved.

Figs. 9 and 10 show comparisons of maximum tensile stress and maximum compressive stress, respectively, versus position along the length of Pile I (point free) for various time increments. Figs. 11 and 12 show comparisons of maximum tensile stress and maximum compressive stress, respectively, versus position along the length of Pile I (point fixed) for ten, twenty, and forty segments.

Effect of Gravity.—The procedure as presented by Smith does not account for the static weight of the pile. In other words, at $t = 0$, all springs, both internal and external, exert zero force. Stated symbolically, $F(m_0) = R(m,0) = 0$. If the effect of gravity is to be included, these forces must be given initial values to produce equilibrium of the system. Strictly speaking, these initial values should be those in effect as a result of the previous blow.

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Not only would it be awkward to "keep books" on the pile throughout the driving so as to identify the initial conditions for successive blows, but it is highly questionable that this refinement would be justified in light of other uncertainties that exist.

Two methods have been developed as a means of entering the gravity effect into the computations. The first of these has been proposed by Smith,\textsuperscript{20} the second has been developed by the writers. Both will be described and compared.

Smith's Approximate Method.—Smith suggests that the external (soil) springs be assumed to resist the static weight of the system according to the relationship

\[ R(m,0) = \frac{Ru(m)}{Ru(\text{total})} \cdot W(\text{total}) \] \hspace{1cm} (12)

in which \( W(\text{total}) \) is the total static weight resisted by soil, in pounds; and \( Ru(\text{total}) \) is the total ultimate ground resistance, in pounds. The quantity \( W(\text{total}) \) is found by

\[ W(\text{total}) = W(b) + F(c) + \sum_{m=2}^{m=p} W(m) \] \hspace{1cm} (13)

in which \( W(b) \) is the weight of the body of the hammer, excluding ram, in pounds; and \( F(c) \) is the force exerted by compressed gases, as under the ram of a diesel hammer, in pounds. The internal forces that initially exist in the pile may now be obtained:

\[ F(1,0) = W(b) + F(c) \] \hspace{1cm} (14)

and in general,

\[ F(m,0) = F(m-1,0) + W(m) - R(m,0) \] \hspace{1cm} (15)

In the absence of compressed gases and hammer weight resting on the pile system, the right side of Eq. 14 is zero. The amount that each internal spring \( m \) is compressed may now be expressed as

\[ C(m,0) = \frac{F(m,0)}{K(m)} \] \hspace{1cm} (16)

The displacement of the point may be obtained from

\[ D(p,0) = \frac{R(p,0)}{K'(p)} \] \hspace{1cm} (17)

By working progressively upward from the point, other displacements are determined from

\[ D(m,0) = D(m+1,0) + C(m,0) \] \hspace{1cm} (18)

\textsuperscript{20} Personal correspondence dated April 13, 1961, from E. A. L. Smith to Charles H. Samson, Jr.
For the inclusion of the gravity effect, Eq. 7 should be modified as follows:

\[ V(m, t) = V(m, t - 1) + \left[ F(m-1, t) - F(m, t) - R(m, t) + W(m) \right] \frac{\Delta t}{W(m)} \quad \ldots \quad (19) \]

In order that the initial conditions of the external springs be compatible with the assumed initial forces \( R(m, 0) \) and initial displacements \( D(m, 0) \), plastic displacements \( D'(m, 0) \) should be set equal to \( D(m, 0) - R(m, 0)/K'(m) \). Influence-coefficient Methods: A second approach that may be used is to treat the system as a statically indeterminate structure. The assumption is made that all springs behave as though linearly elastic in supporting the static weight. The influence-coefficient method\(^{21}\) will be determined here as one means of solving the problem. Fig. 13 illustrates the two parts to the solution. The particular solution is obtained for the statically determinate structure (primary structure) formed by cutting all external springs except the last and by loading with the applied loads \( W_1, W_2, \ldots \) In this particular discussion, subscripts will be used as a convenience in place of the functional designation used in other sections of the paper. The displacements for the various elements of this loaded structure are denoted by \( D_{10}, D_{20}, \ldots \) \( D_{m0}, \ldots \) \( D_{p0} \). In the formulation of the complementary solution, influence coefficients \( d_{11}, d_{12}, \ldots d_{np} \) are generated by successively applying unit loads (positive downward) at points 1, 2, \ldots \( p \) on the primary structure. For example, \( d_{12} \) is the deflection at point 1 caused by a unit load at point 2. Corresponding displacements caused by \( R_1, R_2, \ldots R_p \) can now be expressed in terms of influence coefficients. For example, the complete equation for the actual displacement at a point \( m \) may be stated as

\[ D_m = D_{m0} - d_{m1}R_1 - d_{m2}R_2 - \ldots - d_{mm}R_m - d_{mp}R_p \quad \ldots \quad (20) \]

By considering the external spring at \( m \) as a free body, the following is obtained:

\[ D_m = \frac{R_m}{K_m'} \quad \ldots \quad (21) \]

Substituting from Eq. 21 into Eq. 20 produces

\[ \frac{R_m}{K_m'} = D_{m0} - d_{m1}R_1 - d_{m2}R_2 - \ldots - d_{mm}R_m - d_{mp}R_p \quad \ldots \quad (22) \]

Similar equations may be obtained for the other points of redundancy.

Using matrix notation for a concise representation of the series of equations typified by Eq. 22 yields

\[ \begin{bmatrix} [d] & -[1/K'] \end{bmatrix} \begin{bmatrix} [R] \end{bmatrix} = \begin{bmatrix} D_0 \end{bmatrix} \quad \ldots \quad (23) \]

in which

\[
[d] = \begin{bmatrix}
d_{11} & d_{12} & \cdots & d_{1p} \\
d_{21} & d_{22} & \cdots & d_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
d_{p1} & d_{p2} & \cdots & d_{pp}
\end{bmatrix}
\]

thus,

\[
\{R\} = \left( [d] + \left[ \frac{1}{K'} \right] \right)^{-1} D_0
\]

(24)

(25)

Of course, any method for solving the set of simultaneous equations implicit in Eq. 23 is satisfactory. However, the matrix formulation of Eq. 28 is quite suitable for digital-computer solution.

The solutions for \([D_0]\) and \([d]\) are both associated with a statically determinate system [see Figs. 13(b) and 13(c)]. They may be found by the following expressions:

\[
D_{p0} = \frac{1}{K_p} \sum_{i=1}^{i=p} W_i
\]

(29)

\[
D_{m0} = D_{m+1,0} + \frac{1}{K_m} \sum_{i=1}^{i=m} W_i
\]

(30)

It is seen that computations may proceed from the point upward for all displacement quantities for the particular and complementary solutions. For simplification, \(i\) is considered to begin with \(i = 1\). This, of course, must be adjusted to a particular pile problem to exclude the ram and any other elements not having side springs.

At this stage, returning to the previous notation, all \(R(m,0)\) values may be established by Eq. 28. Furthermore, by Eq. 21 all \(D(m,0)\) values may be obtained. Compressions \(C(m,0)\) may be determined by

\[
C(m,0) = D(m+1,0) - D(m,0)
\]

(32)

Forces \(F(m,0)\) are now found from

\[
F(m,0) = C(m,0)K(m)
\]

(33)

Eq. 19 again provides the modified general velocity equation and computations may now proceed in the usual manner.

Pile II Studies. — Pile II is identified in Fig. 4. These studies are concerned with the effects of gravity, time interval magnitude, and segment length. All calculations associated with this pile were performed on the basis of the following given data: \(Q(m) = 0.1\) in. for all \(m; J(m) = 0.05\) sec per ft for \(m \neq p; J(p) = 0.15\) sec per ft; \(W(ram) = 5,000\) lb; \(W(pile\ cap) = 700\) lb; \(K(cap\ block) = 2 \times 10^6\) lb per in.; \(e(cap\ block) = 0.50;\) and \(V(1,0) = 12.4\) ft per sec. A special pile point weighing 100 lb is assumed. Various cases of loading are identified in Fig. 4. A total ground resistance, \(R_u\) (total, of 200,000 lb) is used in each case. Other details concerning Pile II are given in Fig. 4.
Fig. 14 is a plot of point displacement versus time for Pile II, Case 1. The upper curve represents the solution with the gravity effect ignored. The lower curve represents the solution including gravity effects obtained both by Smith's approximate method and by the influence-coefficient method; the differences could not be plotted. Fig. 15 is a similar plot for Pile II, Case 4. The greatest effects of gravity are in the range of 5% to 10% for the plots shown.

Figs. 16 and 17 are plots of midpoint stress versus time for Pile II, Cases 1 and 4, respectively. The gravity effect on stresses for these two cases is small. It is of interest to note the comparison of $R(m,0)$ values used in considering Pile II, Case 4. Table 1 shows these values for gravity excluded, for Smith's approximate solution and for the influence-coefficient solution.

Table 2 shows the effect of gravity on maximum compressive force, maximum tensile force, and the permanent set for all five cases of soil resistance for Pile II. The solutions with gravity included were obtained by the influence-coefficient method; however, they are quite similar to those found by Smith's approximate method.

### Table 1 — $R(m,0)$ Values

<table>
<thead>
<tr>
<th>$m$</th>
<th>Gravity excluded</th>
<th>Smith's approximation</th>
<th>Influence-coefficient solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>610</td>
<td>679.91</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>610</td>
<td>651.70</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>610</td>
<td>592.73</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>610</td>
<td>512.87</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>610</td>
<td>600.27</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>610</td>
<td>591.28</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>610</td>
<td>585.41</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>610</td>
<td>582.39</td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>610</td>
<td>582.06</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>610</td>
<td>584.40</td>
</tr>
</tbody>
</table>

Table 3 provides a comparison of maximum compressive forces and permanent set for various magnitudes of time intervals. For this case, the variation in results over the range of time intervals considered is small. For Pile II, the $(\Delta t)_{cr}$ for a segment length of 10 ft (ten segments) is 1/1,667 sec. All results thus far discussed for Pile II were obtained using a segment length of 10 ft.

Table 4 provides a comparison of maximum compressive forces at different positions along the pile for different magnitudes of time intervals and different segment lengths. The results apply to Pile II, Case 2. It is seen that the forces compare closely at a given point on the pile.

**Pile III Studies** — The different resistance cases considered for Pile III are defined in Fig. 4. These studies illustrate the influence of pile driver, cushion block, and pile characteristics on pile behavior. The following data are used:

- Ram Weight: $W(A) = 4,850$ lb; $W(B) = 9,300$ lb; and $W(C) = 14,000$ lb. In the case of ram A, two possibilities are examined: in the first it is assumed that no explosive pressure exists beneath the ram; in the second it is assumed that an explosive pressure of 158,700 lb is present. This simulates a pressure that might be encountered in certain types of diesel hammers.
TABLE 2.—EFFECT OF GRAVITY ON MAXIMUM FORCES AND PERMANENT SET: PILE II

<table>
<thead>
<tr>
<th>Case no.</th>
<th>Maximum Compressive Force, in pounds</th>
<th>Maximum Tensile Force, in pounds</th>
<th>Permanent Set, in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gravity neglected</td>
<td>Gravity included</td>
<td>Gravity neglected</td>
</tr>
<tr>
<td>1</td>
<td>405,400</td>
<td>411,500</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>254,800</td>
<td>290,000</td>
<td>10,300</td>
</tr>
<tr>
<td>3</td>
<td>224,800</td>
<td>297,700</td>
<td>30,300</td>
</tr>
<tr>
<td>4</td>
<td>294,100</td>
<td>296,800</td>
<td>55,000</td>
</tr>
<tr>
<td>5</td>
<td>299,900</td>
<td>301,700</td>
<td>48,800</td>
</tr>
</tbody>
</table>

TABLE 3.—COMPARISON OF MAXIMUM COMPREHENSIVE FORCES AND PERMANENT SET FOR VARIOUS TIME INTERVAL MAGNITUDES: PILE II, CASE 1

<table>
<thead>
<tr>
<th>Δt, in seconds</th>
<th>Maximum compressive force at head, in pounds</th>
<th>Maximum compressive force at midpoint, in pounds</th>
<th>Maximum compressive force at tip, in pounds</th>
<th>Permanent set, in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4,000</td>
<td>289,500</td>
<td>300,800</td>
<td>405,300</td>
<td>0.2031</td>
</tr>
<tr>
<td>1/8,000</td>
<td>289,300</td>
<td>301,200</td>
<td>406,600</td>
<td>0.2049</td>
</tr>
<tr>
<td>1/12,000</td>
<td>289,400</td>
<td>301,400</td>
<td>407,100</td>
<td>0.2055</td>
</tr>
<tr>
<td>1/16,000</td>
<td>285,400</td>
<td>301,400</td>
<td>407,200</td>
<td>0.2058</td>
</tr>
<tr>
<td>1/20,000</td>
<td>289,400</td>
<td>301,400</td>
<td>407,300</td>
<td>0.2060</td>
</tr>
<tr>
<td>1/50,000</td>
<td>289,400</td>
<td>301,400</td>
<td>407,400</td>
<td>0.2064</td>
</tr>
<tr>
<td>1/100,000</td>
<td>288,400</td>
<td>301,400</td>
<td>407,400</td>
<td>0.2066</td>
</tr>
</tbody>
</table>

TABLE 4.—COMPARISON OF MAXIMUM COMPREHENSIVE FORCES FOR VARIOUS SEGMENT LENGTHS AND TIME INTERVAL MAGNITUDES: PILE II, CASE 2

<table>
<thead>
<tr>
<th>Distance from top of pile, in feet</th>
<th>Maximum Compressive Force, in Pounds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10 segments Δt = 1/4,000 second</td>
</tr>
<tr>
<td></td>
<td>100 segments Δt = 1/50,000 second</td>
</tr>
<tr>
<td>5</td>
<td>294,800</td>
</tr>
<tr>
<td>15</td>
<td>288,000</td>
</tr>
<tr>
<td>25</td>
<td>283,700</td>
</tr>
<tr>
<td>35</td>
<td>278,400</td>
</tr>
<tr>
<td>45</td>
<td>272,500</td>
</tr>
<tr>
<td>55</td>
<td>266,600</td>
</tr>
<tr>
<td>65</td>
<td>259,700</td>
</tr>
<tr>
<td>75</td>
<td>253,600</td>
</tr>
<tr>
<td>85</td>
<td>247,700</td>
</tr>
<tr>
<td>95</td>
<td>236,500</td>
</tr>
</tbody>
</table>
Capblock. - \( A_{\text{capblock}} = 240 \times 10^5 \text{ lb per in.} \) and \( e_{\text{capblock}} = 0.50 \).

Pile Cap. - One pile is considered for all solutions. It is assumed to be rigid with \( W_{\text{pile cap}} = 1,150 \text{ lb} \).

Cushion Block. - Two cushion blocks are considered, both assumed weightless. Stiffnesses are specified as follows: \( K(A) = 16.7 \times 10^6 \text{ lb per in.} \); \( K(B) = 167 \times 10^6 \text{ lb per in.} \); and \( e_{\text{cushion block}} = 1.00 \).

Pile. - \( \Delta L = 10 \text{ ft} \) for each of 9 segments. Unless otherwise specified, a modulus of elasticity of 4.95 \( \times 10^6 \text{ psi} \) is assumed.

Soil. - \( f(m) = 0.05 \text{ sec per ft} \) for \( m \neq p \); \( f(p) = 0.15 \text{ sec per ft} \); and \( Q(m) = 0.10 \text{ in. for all m} \).

Figs. 18, 19, 20, and 21 show plots of maximum tensile stress developed in the pile versus ram velocity for different ram weights and different cushion blocks. The results for ram A are given with and without explosive pressure. Similar plots of maximum compressive stress developed in the pile versus ram velocity are provided in Figs. 22, 23, 24, and 25.

Although this study is of limited scope, it illustrates a way in which generalized information can be presented. It is also of interest to appraise the particular results obtained for the ranges of variables treated.

For a given ram, as might be expected, both tensile and compressive stresses usually increase with velocity.

In most cases, the effect of the stiffer cushion block is to produce higher tensile stresses. Minor exceptions may be noted for ram C in Figs. 18, 19, and 20. The influence of cushion-block stiffness on development of tensile stress is quite pronounced in some cases. For example, the stiffer cushion block causes 59% greater tension velocity of 17.8 ft per sec, and Case 1 soil resistance.

The effect of explosive pressure for the ranges of variables considered does not appear marked. For example, for the Case 1 soil resistance, the soft cushion block, and ram velocity of 17.8 ft per sec, the explosive pressure produces a decrease of 7% in the maximum tensile stress developed.

In order to gain information concerning the effect of modulus of elasticity of the pile material, the combination of Case 1 soil resistance, ram A with no explosive pressure, and stiff and soft cushion blocks are considered for moduli of elasticity of 2.5 \( \times 10^6 \text{ psi} \) and 7.5 \( \times 10^6 \text{ psi} \), in addition to the value of 4.95 \( \times 10^6 \text{ psi} \) used in previous calculations. Figs. 26 and 27 illustrate the results in the form of maximum tensile stress and maximum compressive stress, respectively, versus ram velocity. At all velocities considered, both tensile and compressive stresses increase with increase in modulus of elasticity of the pile material.

Comparison of Theoretical and Field Test Results; Pile IV Studies. - Pile IV was field tested during the construction of the Nueces Bay Causeway at Corpus Christi, Tex. Stresses were recorded on a high-speed recording oscillograph. The variation of stress with time at a point 9.5 ft from the head of the pile is plotted in Figs. 28, 29, and 30 as a basis for comparison with theory. The pile had penetrated 45 ft into a soft marine silty-clay.

The theoretical solutions for all three figures were obtained with the following specified information: \( W_{\text{ram}} = 4,850 \text{ lb} \) and \( V_{\text{ram,0}} = 13.8 \text{ ft per sec} \). The gravity effect of the pile was neglected.

The differences in the theoretical solutions result from the manner of accounting for internal damping in the pile. In the theoretical solution of Fig. 28, the pile material is considered to be perfectly elastic. In the solution
shown in Fig. 29, a load-deformation relationship of the form of Fig. 2(b) is assumed in which the coefficient of restitution is 0.8. Although there is no particular justification for this specific shape, it does serve to provide a different internal load-deformation behavior by which internal damping occurs.

![Graph](image)

**FIG. 32.—SEGMENT OF A TYPICAL LOAD-DEFORMATION CURVE OBTAINED BY EQ. 34**

Smith\(^5\) has proposed another means of accounting for internal damping. In place of Eq. 5 Smith suggests

\[
 F(m,t) = C(m,t)K(m) + B(m)K(m) \frac{C(m,t) - C(m,t-1)}{12 \Delta t} \quad \ldots \quad (34)
\]

in which \(B(m)\) is a damping constant for internal spring \(m\). Smith states that a value of \(B\) of approximately 0.0002 in. sec per ft will produce a narrow

![Graph](image)

**FIG. 33.—COMPUTER-PLOTTED VALUES OF STRESS VERSUS TIME INTERVAL**
hysteresis loop. Of course, the value can be adjusted to approximate a known material behavior. By using Eq. 34 and a value of \( B = 0.0016 \) in. per sec per ft, the theoretical curve of Fig. 30 is obtained. The load-deformation diagram is shown in Fig. 2(c). By comparing the theoretical solutions with test results, a reasonably good correlation is seen over some ranges. In light of the limited information available concerning the dynamic behavior of soil, the writers consider the correlations to be encouraging. In order to compare directly the three theoretical solutions, Fig. 31 is provided.

A series of hysteresis loops taken from a problem involving Eq. 34 is shown in Fig. 32. At the present stage of development, Eq. 34 appears to offer the most realistic means of simulating the internal damping characteristics. The load-deformation relationship of Fig. 2(d) has been suggested by Smith as another possible means of accounting for internal damping. However, no solutions have been obtained using this type of diagram.

*Automatic Plotting by Computer.*—With solutions of the type obtained by the wave-theory program, graphical representations of results are often desirable. Fig. 33 is provided in order to illustrate the automatic plotting capability of the digital computer. The particular plot given corresponds to the theoretical solution of Fig. 28. A dashed line has been sketched through the asterisk points provided by the computer.

**CONCLUSIONS**

Although the investigation was not intended to provide a comprehensive parameter study, it is possible to make certain observations with respect to the data presented, as long as the scope of such observations is restricted to the ranges of variables considered.

On the basis of the present investigation, the writers have formed the following conclusions:

1. In the studies presented, good numerical accuracy was obtained by dividing piles into at least ten segments with no segment length exceeding 10 ft and keeping the time interval less than one-half the critical value. For relatively short piles, the number of segments can likely be decreased. It should also be noted that the maximum time interval may be controlled by conditions such as a steel ram striking a stiff capblock.

2. For the cases considered, the gravity effect does not appear severe as far as compressive stresses and permanent sets are concerned. Its influence on tensile stresses can prove significant, however, for prestressed concrete piles.

3. For the limited number of cases in which the gravity effect was considered, there were no significant differences obtained by Smith's approximate method or the influence-coefficient method. However, a much wider range of conditions needs to be studied before any general conclusion can be reached.

4. The influence of explosive pressure on results does not appear pronounced.

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ACKNOWLEDGMENTS

The wave-theory computer program and the field-test data used in this investigation were obtained as part of Research Projects RP-23 and RP-27, respectively, sponsored by the Bridge Division of the Texas Highway Department. Both projects were performed by the Texas Transportation Institute, A. & M. College of Texas. The writers gratefully acknowledge the assistance of Farland C. Bundy, Supervising Design Engineer, Bridge Division, Texas Highway Department, who worked closely with the writers in accomplishing both projects, and D. D. Drew, Engineering Computation Specialist at the Data Processing Center, A. & M. College of Texas, who directed and performed much of the programming. The writers also thank E. A. L. Smith, formerly Chief Mechanical Engineer for Raymond International Inc., and now retired, who maintained a continuing interest throughout the work and who contributed significantly to the accomplishments of the research.

This work has served as the basis for a current project being sponsored jointly by the Bridge Division of the Texas Highway Department and the Bureau of Public Roads.

APPENDIX.—NOTATION

The following symbols have been adopted for use in this paper:

\[ m \] = element number;
\[ p \] = number of \( m \) at point of pile;
\[ f(m) \] = quake of external spring \( m \), in inches;
\[ R(m, t) \] = force exerted by external spring \( m \) on element \( m \) in time interval \( t \) in pounds;
\[ Ru(m) \] = ultimate ground resistance for external spring \( m \), in pounds;
\[ t \] = time, or time interval;
\[ u \] = displacement of a bar cross section in \( x \) direction;
\[ V(m, t) \] = velocity of element \( m \) in time interval \( t \), in feet per second;
\[ W(m) \] = weight of element \( m \), in pounds;
\[ x \] = direction of longitudinal axis of a bar;
\[ \rho \] = mass per unit volume;
\[ \Delta L \] = length of segment;
\[ \Delta t \] = size of time interval, in seconds;
\[ (\Delta t)_{cr} \] = size of critical time interval, in seconds;
\[ (\cdot) \] = functional designation;
\[ [\cdot] \] = rectangular matrix;
\[ \{\cdot\} \] = column matrix; and
\[ [\cdot]^{-1} \] = inverse of a matrix.