2.15 FURTHER DISCUSSION OF DEFORMATIONS UNDER AXIAL LOADING; RELATION AMONG $E$, $\nu$, AND $G$

$$P = \tau_{xy}A = (1800 \text{ psi})(8 \text{ in.})(2.5 \text{ in.}) = 36.0 \times 10^3 \text{ lb}$$
$$P = 36.0 \text{ kips}$$

$$\tau = \frac{V}{A}$$

$${\frac{E}{2G}} = 1 + \nu$$

Fig. 2.47

Fig. 2.48

Fig. 2.49 Representations of strain in an axially-loaded bar.
\[ \gamma = \frac{\delta v}{R} \quad \tau = \frac{V}{A} \]

\[ \text{Slope} = \frac{\tau}{\gamma} = G \]

\[ \sigma_{\text{all}} = \tau \]  
\[ T_{\text{all}} = 0.6 \sigma_{\text{all}} \]
\[ T_y = 0.6 T_y \]
\[ T_u = 0.6 T_u \]

\( \gamma \) = shear angle = shear strain  
\( \tau \) = shear stress  
\( G \) = shear modulus  
or modulus of rigidity
2.18 STRESS CONCENTRATIONS

defines the ratio

\[ \sigma_{\text{max}} = K \sigma_{\text{avg}} \]

\[ K = \frac{\sigma_{\text{max}}}{\sigma_{\text{avg}}} \]  \hspace{1cm} (2.48)

defines the ratio of the maximum stress over the average stress computed in the

\[ \sigma_{\text{avg}} = \frac{1}{A} \int_{0}^{A} \sigma(x) \, dx \]

of the maximum stress over the average stress computed in the

\[ \sigma_{\text{max}} \]

of the section. This is defined

Fig. 2.58 Stress distribution near circular hole in flat bar under axial loading.

Fig. 2.59 Stress distribution near fillets in flat bar under axial loading.
(a) Flat bars with holes

**Fig. 2.60** Stress concentration factors for flat bars under axial loading

Note that the average stress must be computed across the narrowest section: $\sigma_{av} = \frac{P}{t d}$, where $t$ is the thickness of the bar.

(b) Flat bars with fillets

shown in Fig. 2.60. To determine the maximum stress occurring near a

---

**EXAMPLE 2.12**

Determine the largest axial load $P$ that can be safely supported by a flat shaft.

---

\[ \text{W. D. Pilkey, Peterson's Stress Concentration Factors, 2nd ed., John Wiley & Sons, New York, 1997.} \]
(a) Flat bars with holes
(b) Flat bars with fillets
\[ T = \frac{T_c}{J} \]

Graph with axes labeled as follows:
- Y-axis: 2.0, 1.9, 1.8, 1.7, 1.6, 1.5, 1.4, 1.3, 1.2, 1.1
- X-axis: 0.00, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30
- Ratio of D/d: 2.5, 2.0, 1.67, 1.25, 1.11

Text:
- \[ T_{avg} = \frac{T_c}{J} \] (Small Shaft)
Appendix C. Properties of Rolled-Steel Shapes
(U.S. Customary Units)

**S Shapes**
(American Standard Shapes)

<table>
<thead>
<tr>
<th>Designation</th>
<th>$A$, in$^2$</th>
<th>$d$, in.</th>
<th>Width $b_1$, in.</th>
<th>Flange Thickness $t_{f}$, in.</th>
<th>Web Thickness $t_{w}$, in.</th>
<th>$I_y$, in$^4$</th>
<th>$S_y$, in$^3$</th>
<th>$x_y$, in.</th>
<th>$I_z$, in$^4$</th>
<th>$S_z$, in$^3$</th>
<th>$r_y$, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S24 × 121</td>
<td>35.6</td>
<td>24.50</td>
<td>8.050</td>
<td>1.090</td>
<td>0.800</td>
<td>3160</td>
<td>258</td>
<td>9.43</td>
<td>83.3</td>
<td>20.7</td>
<td>1.53</td>
</tr>
<tr>
<td>106</td>
<td>31.2</td>
<td>24.50</td>
<td>7.870</td>
<td>1.090</td>
<td>0.620</td>
<td>2940</td>
<td>240</td>
<td>9.71</td>
<td>77.1</td>
<td>19.6</td>
<td>1.57</td>
</tr>
<tr>
<td>100</td>
<td>29.3</td>
<td>24.00</td>
<td>7.245</td>
<td>0.870</td>
<td>0.745</td>
<td>2390</td>
<td>199</td>
<td>9.02</td>
<td>47.7</td>
<td>13.2</td>
<td>1.27</td>
</tr>
<tr>
<td>90</td>
<td>26.5</td>
<td>24.00</td>
<td>7.125</td>
<td>0.870</td>
<td>0.625</td>
<td>2250</td>
<td>187</td>
<td>9.21</td>
<td>44.9</td>
<td>12.6</td>
<td>1.30</td>
</tr>
<tr>
<td>80</td>
<td>23.5</td>
<td>24.00</td>
<td>7.000</td>
<td>0.870</td>
<td>0.500</td>
<td>2100</td>
<td>175</td>
<td>9.47</td>
<td>42.2</td>
<td>12.1</td>
<td>1.34</td>
</tr>
</tbody>
</table>
### Properties of Rolled-Steel Shapes (U.S. Customary Units)

#### W Shapes (Wide-Flange Shapes)

<table>
<thead>
<tr>
<th>Designation</th>
<th>Area, in²</th>
<th>Depth, in</th>
<th>Flange</th>
<th>Web</th>
<th>Axis X-X</th>
<th>Axis Y-Y</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A, in²</td>
<td>d, in</td>
<td>Width, in</td>
<td>Thickness, in</td>
<td>I_x, in⁴</td>
<td>S_x, in³</td>
</tr>
<tr>
<td>W12 x 96</td>
<td>28.2</td>
<td>12.71</td>
<td>12.160</td>
<td>0.900</td>
<td>0.550</td>
<td>12.040</td>
</tr>
<tr>
<td>72</td>
<td>21.1</td>
<td>12.25</td>
<td>8.080</td>
<td>0.640</td>
<td>0.370</td>
<td>974</td>
</tr>
<tr>
<td>50</td>
<td>14.7</td>
<td>12.19</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[
P = \frac{P}{A} = \frac{V}{KSI}
\]

\[
\epsilon = \frac{S}{L} = \frac{IN}{IN}
\]

\[
E = \frac{S}{\epsilon} = \frac{P}{A} = \frac{PL}{AS}
\]

\[
S = \frac{PL}{AE}
\]
Page 21

Pop Quiz

Q.D. = 2\text{1/8}''

Wall thickness = \frac{1}{8}''

Load = 127\text{K}

L = 16''

STEEL

\[ S = \? \]

\[ G = \? \]

\[ A = \frac{\pi}{4} \left( 2.125^2 - (2.125^2 - \frac{1}{8}''^2) \right) \]

\[ S = \frac{(127\text{K})(16'')}{A (29,000 \text{KIN}^2)} \]

\[ \sigma = 127\text{K} / A \]

\[ \sigma_c = \sigma_{AB} + \sigma_{BC} = \frac{P_{AB} L_{AB}}{A_{AB} E_{AB}} + \frac{P_{BC} L_{BC}}{A_{BC} E_{BC}} \]

\[ = \frac{(-20\text{KN})(6m)}{(0.01\text{m}^2)(200\times10^6\text{KN/m}^2)} + \frac{(-80\text{KN})(4m)}{(0.02\text{m}^2)(73\times10^6\text{KN/m}^2)} \]

\[ = \]
\( \sigma_{AB} = \frac{P_{AB}}{A_{AB}} = \frac{-20 \text{ kN}}{0.01 \text{ m}^2} = \) (comp)

\( \sigma_{BC} = \frac{P_{BC}}{A_{BC}} = \frac{-80 \text{ kN}}{0.02 \text{ m}^2} = \) (TENS)

\[
S = \int \frac{P_x \, dx}{A_x \, E_x} = \frac{P}{E} \int \frac{dx}{A_x}
\]

\[
S = \frac{1}{EA} \int (Py) \, dy = \frac{1}{EA} \int (Py) \, dy
\]

\[Py = \text{Volume} \times \text{unit weight} = A \times y \times \text{unit weight}\]

\( \sigma_{\text{steel}} = 0.293 \text{ kN/} \text{m}^3 \)

\( S = \frac{1}{E} \int (A \times y \times \text{unit weight}) \, dy = \frac{x}{E} \left[ \frac{y^2}{2} \right]_0^L = \frac{xL^2}{2E} \)
\[ \sigma_c = \frac{P_{AB} L_{AB}}{A E} + \frac{Q L_{BC}}{A E} \]

\[ \sigma_A = \frac{Q L_{AB}}{A E} + \frac{P L_{BC}}{A E} \]

\[ \sum F_H = 0 = -P_{AB} + P - P_{BC} \]

\[ \sigma_{AB} = \frac{P_{AB} L_{AB}}{A E} \]

\[ \sigma_{BC} = \frac{P_{BC} L_{BC}}{A E} \]

\[ \sigma_{AB} = \sigma_{BC} \]

\[ \frac{P_{AB} L_{AB}}{A E} = \frac{P_{BC} L_{BC}}{A E} \]

\[ \tau_{AB} = \frac{P_{AB}}{A} \]
\[ \sigma_c = \sigma_{c1} - \sigma_{c2} = 0 \]
Temperature Deformations

\[ \varepsilon_T = \alpha \Delta T L \]

\[ \alpha = \frac{\varepsilon}{\Delta T L} \]

Steel bar:
\[ L = 6 \text{ ft} \]
\[ \Delta T = +400 \degree F \]
\[ \varepsilon_T = 0.0156 \text{ ft} \]

6 ft steel bar
\[ \Delta T = 400 \degree F \text{ heat} \]
What is load in bar?

\[ \varepsilon_{\text{end}} = +\alpha \Delta T L - \frac{EL}{AE} = 0 \]

\[ F = \frac{\alpha \Delta T A L}{AE} \]
Poisson's Ratio

\[ \varepsilon_{\text{long}} = \frac{\sigma_{\text{long}}}{L_{\text{long}}} \]

\[ \varepsilon_{\text{lat}} = \frac{\sigma_{\text{lat}}}{L_{\text{lat}}} \]

\[ \nu = - \frac{\varepsilon_{\text{lat}}}{\varepsilon_{\text{long}}} \]
What is $S_D$?

$S_B = \frac{P_{BE} L_{BE}}{A_{BE} E_{BE}}$
Solutions to Pop Quizzes (Thanks P.I.)

Solve for stress and deflection in the following pipe:

**POP QUIZ**

\[ \varnothing \quad OD = 2\frac{3}{8}'' \]

Wall thickness \[ \varphi = 1/8'' \]

You weigh \( 127 \text{ lbs} = P \)

\( L = 1\text{ ft} \)

\[ \delta = \, ?, \quad \sigma = \, ? \]

\[ \delta = \frac{PL}{AE} = 8.92 \times 10^{-5} \text{ in} \]

\[ \sigma = \frac{P}{A} = 161.7 \text{ psi, compressive} \]

Solve for the temperature deflection due to a rise in temperature:

\[ \Delta T = 400^\circ F \]

\( L = 6\text{ ft} \)

Steel

\( A = 6 \text{ in}^2 \)

\[ \delta_T = a \Delta T \]

\( L = 10' \), \( \Delta T = 100^\circ F \)

\[ \delta_T = 0.0152\text{ in} \]
What is $S_D$?

\[ \Sigma M_A = 0 = -100 \text{kip} \times 10 \text{ft} + F_{BE} \times 6 \text{ft} \]

\[ F_{BE} = 166.7 \text{kip} \]

\[ S_{BE} = \frac{F_{BE} \times L_{BE}}{A_{BE} \times E_{BE}} = \frac{166.7 \text{kip} \times 5 \text{ft} \times 12 \text{in/ft} \times 12 \text{in}}{3 \text{in}^2 \times 10.6 \times 10^3 \text{kip/ft}^2} = 0.314 \text{in} \]

\[ \frac{S_{BE}}{6 \text{ft}} = \frac{S_D}{18 \text{ft}} \]

\[ 2 \times S_D = \frac{18 \text{ft}}{6 \text{ft}} \times 0.314 \text{in} = 0.9431 \text{in} = 0.0786 \text{ft} \]
What is the maximum shear strain in the steel bar shown? Bar is 2" x 4" x 36" long.

- Force: $P = 160 \text{kN}$
- Area: $A = 2" \times 4" = 8 \text{in}^2$
- $T = \frac{P}{A} \sin \theta \cos \theta$

**Equation (1.16)**

$$T_{\max} = \frac{P}{2A} = \frac{160 \text{kN}}{2(2" \times 4")} = 10 \text{kN/in}^2$$

**Equation (2.36)**

$$\varepsilon_{\max} = \frac{T}{G} = \frac{10 \text{kN/in}^2}{11.2 \times 10^3 \text{kN/m}^2} = 8.92 \times 10^{-6} \text{in/in}$$

**Page A12**

- **Strain**
  - $\varepsilon = \frac{P_L}{AE}$
  - $\varepsilon = \frac{P}{AE}$
\[ dT = T dA \]

\[ T = \int T dA \quad \text{Eq. (0)} \]

Plane sections remain plane

Assume Wall to \( A = L \) before bar twisted
Assume Wall to \( A' > L \) after twisting

If so, then Wall to \( C \) also would have to lengthen same amount.
So, if true, all points would have to lengthen & plane section \( AC \) would have to move to \( AC' \), so:

Plane sections remain plane after \( T \) is applied.
Remember:

\[ T_{xy} = T_{yx} \]

\[ \phi = \text{angle of twist} \]
\[ @ \text{end of bar} \]

\[ \tau = \text{shearing angle} \]

\[ \tan \tau = \frac{AB}{L} \quad \text{or} \quad \tau = \frac{AB}{L} \quad \text{so} \quad AB = \tau L \]

\[ \tan \phi = \frac{AB}{\rho} \quad \text{or} \quad \phi = \frac{AB}{\rho} \quad \text{so} \quad AB = \phi \rho \]

\[ \tau L = \phi \rho \]

\[ \text{Eq}(1) \quad \phi = \frac{\text{any}_{\rho}}{L} \text{ anywhere inside bar} \]

\[ \text{and} \quad \tau = \frac{\phi}{\rho} \text{ on outside radius of bar} \]

\[ \text{Eq}(2) \quad \text{max} L \text{ when } \rho = C \]

\[ \text{Eq}(1) \quad \phi = \frac{\text{any}_{\rho}}{L} \]

\[ \text{Eq}(2) \quad \phi = \frac{\max_{\rho}}{C} \text{ and} \]

\[ \frac{\text{any}_{\rho}}{C} = \frac{\max_{\rho}}{C} \text{ and } \]

\[ \phi = \frac{\max_{\rho}}{C} \text{ Eq}(3) \]
Since \( T = G \sigma \) \( \frac{G}{E} \)

\[ \frac{T}{G} = \frac{P}{c} \frac{T_{\text{max}}}{G} \]

Substitute into Eq. (3):

\[ \frac{T_{\text{any}}}{G} = \frac{P}{c} \frac{T_{\text{max}}}{G} \]

Eq. (4) so \( T_{\text{anywhere in bar}} = \frac{P}{c} T_{\text{max on outside surface of bar}}. \)

So \( T_{\text{anywhere varies linearly from zero at } \rho = 0 \text{ (center of bar)} \text{ to } T_{\text{max on outside surface of bar at } \rho = c}. \)

Now from Eq. (5):

\[ T = \int_{T_{\text{any}}} \rho dA \]

\[ T = \left( \frac{P}{c} T_{\text{max}} \right) \rho dA \]

Eq. (4a)  \( T = \frac{T_{\text{max}}}{c} \int_{\text{area}} \rho^2 dA \)

Eq. (5)  \( T = \left( T_{\text{max}} \right) \left( \frac{J}{c} \right) \) (\( J \) = polar moment of inertia)
\[ j = \text{POLAR MOMENT OF INERTIA} \]
\[ S = r \theta \]
\[ da = rd\theta = \rho d\theta \]

\[ dA = dp \rho d\theta \]
\[ \int_0^c \int_0^{2\pi} \rho^2 dA = \int_0^c \int_0^{2\pi} \rho^2 dp \rho d\theta = \int_0^c \int_0^{2\pi} \rho^3 dp d\theta \]
\[ = \int_0^c \rho^3 dp (2\pi) \]
\[ = \rho^4 \left[ 2\pi \frac{c}{4} \right] = 2\pi c^2 \frac{\rho^4}{4} = \frac{\pi c^4}{4} = j \]
\[ = \frac{\pi D^4}{32} \]
\[ T_{\text{max}} = \frac{P J}{c} \]

from Eq. (1) pg. 146c

\[ T_{\text{max}} = \frac{P J}{J} = \frac{P J}{J} \]

\[ J \text{ from math 151(?) = } \frac{\pi}{2} (C^4) \text{ for solid} \]

\[ J_{\text{pipe}} = \frac{\pi}{2} (\text{Outer} - \text{Inner})^4 \text{ for pipes} \]

\[ \pi/32 (D_0^4 - D_i^4) \]

OLD Pop Quiz

Solve for \( T_{\text{max}} \) on A36 steel bar painted green loaded as shown. \( c = 1.5" \)

\[ T = 140 \text{ kFt} \]

\[ T = 80 \text{ kFt} \]

\[ T = 60 \text{ kFt} \]

\[ T = \frac{P J}{J} \]
It follows from Eq. (3.2) that the shearing strain is maximum on the surface of the shaft, where \( \rho = c \). We have

\[
\gamma_{\text{max}} = \frac{c \phi}{L} \quad \text{(3.3)}
\]

Eliminating \( \phi \) from Eqs. (3.2) and (3.3), we can express the shearing strain \( \gamma \) at a distance \( \rho \) from the axis of the shaft as

\[
\gamma = \frac{\rho}{c} \gamma_{\text{max}} \quad \text{(3.4)}
\]

\[
\begin{align*}
\tau &= \frac{P}{A} \quad \tau_{\text{max}} = \frac{T}{I_c} \\
\end{align*}
\]

\[
\text{(Eq. 4 pg 146c)} \quad \tau = \frac{\rho}{c} \tau_{\text{max}} \quad \text{(3.6)}
\]

\[
\tau_{\text{min}} = \frac{c_1}{c_2} \tau_{\text{max}} \quad \text{(3.7)}
\]

We now recall from Sec. 3.2 that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude \( T \) of the torque exerted on the shaft:

\[
\int \rho (\tau dA) = T \quad \text{(3.1)}
\]
Substituting for $\tau$ from (3.6) into (3.1), we write

$$Eqs. 4a, pg. 146 \quad T = \int \tau \rho \, dA = \frac{\tau_{\text{max}}}{c} \int \rho^2 \, dA$$

$$J = \frac{1}{2} \pi c_2^4 - \frac{1}{2} \pi c_1^4 = \frac{1}{2} \pi (c_2^4 - c_1^4) \quad (3.11)$$

We note that, if SI metric units are used in Eq. (3.9) or (3.10), $T$ will be expressed in N m, $c$ or $\rho$ in meters, and $f$ in m$^4$; we check that the resulting shearing stress will be expressed in N/m$^2$, that is, pascals (Pa). If U.S. customary units are used, $T$ should be expressed in lb \cdot in., $c$ or $\rho$ in inches, and $f$ in in$^4$, with the resulting shearing stress expressed in psi.
Design the lightest A36 steel shaft to carry 60 kft of torque using a FOS = 2.2 and Tult = 32 ksi.

\[ G = 11.2 \times 10^3 \text{ ksi} \]

\[ T_{\text{max}} = \frac{T_c}{J} = \frac{T_c}{\frac{\pi}{4} C^4} \]

\[ S_a = \frac{C^4}{C} = C^3 = \frac{T_{\text{2}}}{\pi T_{\text{max}}} \]

\[ C = 3.16 \text{ in} \]

\[ D = 6.32 \text{ in} \]

\[ (K \text{ ft}) \text{ in}^2 (12 \text{ in/ft}) \]

\[ K \]
Old Pop Quiz

How large a hole can be drilled into a 6" diameter solid A36 steel shaft and still safely carry 30 kft of torque. Base design on $T_{yield} = 20$ ksi w/ Fos = 2.0

\[ J = \frac{\pi}{2}(R_0^4 - R_i^4) \]

\[ T_{max} = \frac{T_{C, max}}{J} \]

\[ T_{all} = \frac{20^k}{1 N^2 (2.0)} = \frac{(30 kft)(\frac{12}{\text{in}})(3 \text{ in})}{\frac{\pi}{2}(-3 \text{ in})^4 - (R_i^4)} \]

\[ R_i = 1.87 \text{ in} \]
EXAMPLE 3.01

\[ J = \frac{\pi}{2} (c_0^4 - c_i^4) \]

\[ \tau_{\text{max}} = \frac{Tc}{J} \quad T = \frac{J\tau_{\text{max}}}{c} \quad (3.12) \]

Recalling that the polar moment of inertia \( J \) of the cross section is given by Eq. (3.11), where \( c_1 = \frac{1}{2}(40 \text{ mm}) = 0.02 \text{ m} \) and \( c_2 = \frac{1}{2}(60 \text{ mm}) = 0.03 \text{ m} \), we write

\[ J = \frac{1}{2} \pi (c_2^4 - c_1^4) = \frac{1}{2} \pi (0.03^4 - 0.02^4) = 1.021 \times 10^{-6} \text{ m}^4 \]

Substituting for \( J \) and \( \tau_{\text{max}} \) into (3.12), and letting \( c = c_2 = 0.03 \text{ m} \), we have

\[ T = \frac{J\tau_{\text{max}}}{c} = \frac{(1.021 \times 10^{-6} \text{ m}^4)(120 \times 10^6 \text{ Pa})}{0.03 \text{ m}} = 4.08 \text{ kN \cdot m} \]

(b) Minimum Shearing Stress. The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (3.7), which expresses that \( \tau_{\text{min}} \) and \( \tau_{\text{max}} \) are respectively proportional to \( c_1 \) and \( c_2 \):

\[ \tau_{\text{min}} = \frac{c_1}{c_2} \tau_{\text{max}} = \frac{0.02 \text{ m}}{0.03 \text{ m}} (120 \text{ MPa}) = 80 \text{ MPa} \]

Fig. 3.15

Fig. 3.16 Shaft with variable cross section.
3.4 Stresses in the Elastic Range

Since the faces of element $a$ are respectively parallel and

![Diagram of circular shaft with elements at different orientations]

**Fig. 3.17** Circular shaft with elements at different orientations.

$A = \frac{\cos 45^\circ}{\cos 45^\circ} = A_0$

$A = \frac{A_0}{\cos 45^\circ}$

**Fig. 3.18** Forces on faces at $45^\circ$ to shaft axis.

$A = \frac{A_0}{0.707} = A_0 \sqrt{2}$

**Fig. 3.19** Shaft with elements with only shear stresses or normal stresses.

$\tau = \frac{\sqrt{2}}{2} \frac{P_{\text{max}}}{A}$

\[ = \frac{P_A}{A} = \frac{T_{\text{max}}}{J} \]

\[ = \frac{T_c}{J} \]

---

i.e., along surfaces forming a $45^\circ$ angle with the longitudinal axis of the specimen (Photo 3.2b).

\[ = \frac{T_c}{J} \]

\[ = \frac{P_A}{A} \]

\[ = \frac{T_{\text{max}}}{J} \]

\[ = \frac{T_c}{J} \]

---

(a) Ductile failure

**Photo 3.2** Shear failure of shaft subject to torque.

1Stresses on elements of arbitrary orientation, such as element $b$ of Fig. 3.18, will be discussed in Chap. 7.
\[ T = \frac{P}{A} \]
\[ T_{\max} = \frac{F_c}{J} \]
\[ T = \frac{\sigma}{2} \]
\[ T = T_{\max} \]
\[ \Delta \max = \frac{c\phi}{L} \]

Page 148, Eq. 3.3

Definition of \( G \)

\[ \Delta \max = \frac{T_{\max}}{G} \]

so

\[ \frac{c\phi}{L} = \frac{T_{\max}}{G} = \frac{T_c}{J} \cdot \frac{1}{G} \]

so

\[ \phi = \frac{T_c}{GJ} \]

Angle of twist