Chapter 4 Pure Bending

4.1 INTRODUCTION

\[ \gamma = \frac{I}{2 \times c^2} \]

Fig. 4.2 Beam in which portion CD is in pure bending.

Photo 4.1 For the sport buggy shown, the center portion of the rear axle is in pure bending.
Fig. 4.3 Forces exerted on clamp.

Fig. 4.4 Cantilever beam, not in pure bending.
Pure Bending
press that the system of the elementary internal forces exerted on the section is equivalent to the couple \( \mathbf{M} \) (Fig. 4.8).

\[ \sigma = \frac{dF}{dA} \]

\[ \begin{align*}
    \tau_x dA &= \sigma_x dA \\
    \tau_y dA &= \sigma_y dA \\
    \tau_z dA &= \sigma_z dA
\end{align*} \]

\[ F = \int \tau_x dA + \int \tau_y dA + \int \tau_z dA = \int (\sigma_x dA + \sigma_y dA + \sigma_z dA) \]

\[ J = \frac{dF}{dA} \]

\[ \begin{align*}
    J &= \frac{dF}{dA} \\
    I &= \frac{dF}{dA}
\end{align*} \]

\[ \begin{align*}
    \int \sigma_x dA &= 0 \\
    \int \sigma_y dA &= 0 \\
    \int \sigma_z dA &= 0
\end{align*} \]

moments about \( y \) axis: \[ \int \sigma_x dA = 0 = \int \sigma_z dA \] (4.2)

moments about \( z \) axis: \[ \int (-y \sigma_x dA) = +M \] (4.3)

Three additional equations could be obtained by setting equal to zero the sums of the \( y \) components, \( z \) components, and moments about the \( x \) axis, but these equations would involve only the components of the shearing stress and, as you will see in the next section, the components of the shearing stress are both equal to zero.

Two remarks should be made at this point: (1) The minus sign in Eq. (4.3) is due to the fact that a tensile stress \( (\sigma_x > 0) \) leads to a negative moment (clockwise) of the normal force \( \sigma_x dA \) about the \( z \) axis. (2) Equation (4.2) could have been anticipated, since the application of couples in the plane of symmetry of member \( AB \) will result in a distribution of normal stresses that is symmetric about the \( y \) axis.

Once more, we note that the actual distribution of stresses in a given cross section cannot be determined from statics alone. It is statically indeterminate and may be obtained only by analyzing the deformations produced in the member.
(a) Longitudinal, vertical section (plane of symmetry)

(b) Longitudinal, horizontal section

Intersects a transverse section along a straight line called the *neutral axis* of the section (Fig. 4.12b). The origin of coordinates will now be se-

lected on the neutral surface, rather than on the lower face of the member as done earlier, so that the distance from any point to the neutral surface will be measured by its coordinate $y$.

†Also see Prob. 4.38.
Denoting by \( \rho \) the radius of arc \( DE \) (Fig. 4.12a), by \( \theta \) the central angle corresponding to \( DE \), and observing that the length of \( DE \) is equal to the length \( L \) of the undeformed member, we write

\[
DE: \quad L = \rho \theta
\]  
(4.4)

Considering now the arc \( JK \) located at a distance \( y \) above the neutral surface, we note that its length \( L' \) is

\[
JK: \quad L' = (\rho - y)\theta
\]  
(4.5)

Since the original length of arc \( JK \) was equal to \( L \), the deformation of \( JK \) is

\[
\delta = L' - L
\]  
(4.6)

or, if we substitute from (4.4) and (4.5) into (4.6),

\[
\delta = (\rho - y)\theta - \rho \theta = -y\theta
\]  
(4.7)

The longitudinal strain \( \varepsilon_x \) in the elements of \( JK \) is obtained by dividing \( \delta \) by the original length \( L \) of \( JK \). We write

\[
\varepsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho \theta}
\]  
(4.8)

The minus sign is due to the fact that we have assumed the bending moment to be positive and, thus, the beam to be concave upward.

Because of the requirement that transverse sections remain plane, identical deformations will occur in all planes parallel to the plane of symmetry. Thus the value of the strain given by Eq. (4.8) is valid anywhere, and we conclude that the **longitudinal normal strain** \( \varepsilon_x \) **varies linearly with the distance** \( y \) **from the neutral surface**.

The strain \( \varepsilon_x \) reaches its maximum absolute value when \( y \) itself is largest. Denoting by \( c \) the largest distance from the neutral surface (which corresponds to either the upper or the lower surface of the member), and by \( \varepsilon_m \) the **maximum absolute value** of the strain, we have

\[
\varepsilon_m = \frac{c}{\rho} \quad \rho = \frac{c}{\varepsilon_m}
\]  
(4.9)

Solving (4.9) for \( \rho \) and substituting the value obtained into (4.8), we can also write

\[
\varepsilon_x = -\frac{y}{c}\varepsilon_m
\]  
(4.10)

We conclude our analysis of the deformations of a member in pure bending by observing that we are still unable to compute the strain or stress at a given point of the member, since we have not yet located the neutral surface in the member. In order to locate this surface, we must first specify the stress-strain relation of the material used.†

†Let us note, however, that if the member possesses both a vertical and a horizontal plane of symmetry (e.g., a member with a rectangular cross section), and if the stress-strain curve is the same in tension and compression, then the member will compress equally in both directions.
\[ \varepsilon_x = -\frac{\sigma_x}{E} \]

\[ E = \frac{\sigma_x}{\varepsilon} = \frac{\sigma_{\max}}{\varepsilon_{\max}} \]

so \( \varepsilon_x = \sigma_x / E \)

\[ \varepsilon_{\max} = \frac{\sigma_{\max}}{E} \]

\[ \varepsilon_x = \frac{\sigma_x}{E} = -\frac{\sigma_x}{E} \cdot \frac{\sigma_{\max}}{E} \]

\[ \sigma_x = -\frac{\sigma_x}{E} \cdot \sigma_{\max} \quad (4.12) \]

But (Eq. 4.11)

\[ \int \sigma_x dA = 0 \quad \text{Go to eq. 229} \]
its modulus of elasticity, we have in the longitudinal $x$ direction

$$
\sigma_x = E \varepsilon_x = E \left( -\frac{1}{E} \varepsilon_m \right) \quad (4.11)
$$

Recalling Eq. (4.10), and multiplying both members of that equation by $E$, we write

$$
E \varepsilon_x = -\frac{y}{c} (E \varepsilon_m)
$$

or, using (4.11),

$$
\sigma_x = -\frac{y}{c} \sigma_m \quad (4.12)
$$

where $\sigma_m$ denotes the maximum absolute value of the stress. This result shows that, in the elastic range, the normal stress varies linearly with the distance from the neutral surface (Fig. 4.13).

It should be noted that, at this point, we do not know the location of the neutral surface, nor the maximum value $\sigma_m$ of the stress. Both can be found if we recall the relations (4.1) and (4.3) which were obtained earlier from statics. Substituting first for $\sigma_x$ from (4.12) into (4.1), we write

$$
\int \sigma_x \, dA = \int \left( -\frac{y}{c} \sigma_m \right) \, dA = -\frac{\sigma_m}{c} \int y \, dA = 0
$$

from which it follows that

$$
\int y \, dA = 0 \quad (4.13)
$$

This equation shows that the first moment of the cross section about its neutral axis must be zero.† In other words, for a member subjected to pure bending, and as long as the stresses remain in the elastic range, the neutral axis passes through the centroid of the section.

We now recall Eq. (4.3), which was derived in Sec. 4.2 with respect to an arbitrary horizontal $z$ axis,

$$
\mathbf{P_g} \mathbf{225} \cdot \int (-y \sigma_x \, dA) = M \quad (4.3)
$$

Specifying that the $z$ axis should coincide with the neutral axis of the cross section, we substitute for $\sigma_x$ from (4.12) into (4.3) and write

†See Appendix A for a discussion of the moments of areas.
\[ \Sigma dA_i y_i = \int y dA = 0 \]

END VIEWS
\[ I = \frac{\pi c^4}{4} \]

Integrating (4.14) over the cross section, we note that \( I \) is the moment of inertia or second moment, of the cross section with respect to a centroidal axis perpendicular to the plane of the couple \( M \). Solving (4.14) for \( \sigma_m \), we have:

\[ \sigma_m = \frac{Mc}{I} \]

Substituting for \( \sigma_m \) from (4.15) into (4.12), we obtain the normal stress \( \sigma_x \) at any distance \( y \) from the neutral axis:

\[ \sigma_x = -\frac{My}{I} \]

Equations (4.15) and (4.16) are called the elastic flexure formulas, and the normal stress \( \sigma_x \) caused by the bending or "flexing" of the member is often referred to as the flexural stress. We verify that the stress is compressive \( (\sigma_x < 0) \) above the neutral axis \( (y > 0) \) when the bending moment \( M \) is positive, and tensile \( (\sigma_x > 0) \) when \( M \) is negative.

Returning to Eq. (4.15), we note that the ratio \( I/c \) depends only upon the geometry of the cross section. This ratio is called the elastic section modulus and is denoted by \( S \). We have:

\[ S = \frac{I}{c} \]

Substituting \( S \) for \( I/c \) into Eq. (4.15), we write this equation in the alternative form:

\[ \sigma_m = \frac{M}{S} \]

\[ S_{\text{Gallow}} = \frac{M_{\text{Gallow}}}{c} \]

Since the maximum stress \( \sigma_m \) is inversely proportional to the elastic section modulus \( S \), it is clear that beams should be designed with as large a value of \( S \) as practicable. For example, in the case of a wooden beam with a rectangular cross section of width \( b \) and depth \( h \), we have:

\[ S = \frac{I}{c} = \frac{1}{12}bh^3 \frac{1}{h/2} = \frac{1}{6}bh^2 = \frac{1}{6}Ah \]

where \( A \) is the cross-sectional area of the beam. This shows that, of two beams with the same cross-sectional area \( A \) (Fig. 4.14), the beam with the larger depth \( h \) will have the larger section modulus and, thus, will be the more effective in resisting bending.‡

\[ \left( \frac{4\text{in}}{\text{ft}} \right) \left( 30 \text{ft} \right) \left( \frac{44}{12} \right) \left( 20 \right) = 1126 \text{mil} \]

‡We recall that the bending moment was assumed to be positive. If the bending moment was negative, then the stress would be tensile.
\[ I = \frac{M_e}{4} = \frac{M}{H/c} = \frac{M}{S} \]

\[ \frac{I}{d/2} = S_{res} = \frac{50}{1 \text{ in}^3} = \frac{M}{S_{all}} \]

Properties of Rolled-Steel Shapes
(U.S. Customary Units)

<table>
<thead>
<tr>
<th>W Shapes (Wide-Flange Shapes)</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Designation †</th>
<th>Area ( A, \text{in}^2 )</th>
<th>Depth ( d, \text{in.} )</th>
<th>Width ( b, \text{in.} )</th>
<th>Thickness ( t, \text{in.} )</th>
<th>Flange</th>
<th>Web Thickness ( t_w, \text{in.} )</th>
<th>Axis X-X</th>
<th>Axis Y-Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>W12 ( \times 96 )</td>
<td>28.2</td>
<td>12.71</td>
<td>12.160</td>
<td>0.900</td>
<td>0.550</td>
<td>833</td>
<td>131</td>
<td>5.44</td>
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<tr>
<td>35</td>
<td>11.8</td>
<td>11.94</td>
<td>8.005</td>
<td>0.515</td>
<td>0.295</td>
<td>310</td>
<td>51.9</td>
<td>5.13</td>
</tr>
<tr>
<td>30</td>
<td>10.3</td>
<td>12.50</td>
<td>6.560</td>
<td>0.520</td>
<td>0.300</td>
<td>285</td>
<td>45.6</td>
<td>5.25</td>
</tr>
<tr>
<td>26</td>
<td>7.65</td>
<td>12.22</td>
<td>6.520</td>
<td>0.440</td>
<td>0.260</td>
<td>238</td>
<td>38.6</td>
<td>5.21</td>
</tr>
<tr>
<td>22</td>
<td>6.48</td>
<td>12.31</td>
<td>6.490</td>
<td>0.380</td>
<td>0.230</td>
<td>204</td>
<td>33.4</td>
<td>5.17</td>
</tr>
<tr>
<td>16</td>
<td>4.71</td>
<td>11.99</td>
<td>3.990</td>
<td>0.265</td>
<td>0.220</td>
<td>156</td>
<td>25.4</td>
<td>4.91</td>
</tr>
<tr>
<td>19</td>
<td>4.04</td>
<td>10.24</td>
<td>4.020</td>
<td>0.392</td>
<td>0.220</td>
<td>238</td>
<td>50.0</td>
<td>4.66</td>
</tr>
<tr>
<td>15</td>
<td>3.24</td>
<td>9.99</td>
<td>4.000</td>
<td>0.270</td>
<td>0.230</td>
<td>228</td>
<td>52.0</td>
<td>4.64</td>
</tr>
<tr>
<td>12</td>
<td>2.17</td>
<td>8.25</td>
<td>8.070</td>
<td>0.560</td>
<td>0.360</td>
<td>184</td>
<td>43.3</td>
<td>3.61</td>
</tr>
<tr>
<td>18</td>
<td>5.26</td>
<td>8.14</td>
<td>5.250</td>
<td>0.330</td>
<td>0.230</td>
<td>61.9</td>
<td>15.2</td>
<td>3.43</td>
</tr>
<tr>
<td>15</td>
<td>4.44</td>
<td>8.11</td>
<td>4.015</td>
<td>0.315</td>
<td>0.245</td>
<td>48.0</td>
<td>11.8</td>
<td>3.29</td>
</tr>
<tr>
<td>13</td>
<td>3.84</td>
<td>7.99</td>
<td>4.000</td>
<td>0.255</td>
<td>0.230</td>
<td>39.6</td>
<td>9.91</td>
<td>3.21</td>
</tr>
</tbody>
</table>

| W8 \( \times 58 \) | 17.1 | 8.75 | 8.220 | 0.810 | 0.510 | 228 | 52.0 | 3.65 | 75.1 | 18.3 | 2.10 |
| 48 | 14.1 | 8.50 | 8.110 | 0.685 | 0.400 | 184 | 43.3 | 3.61 | 60.9 | 15.0 | 2.08 |
| 40 | 11.7 | 8.25 | 8.070 | 0.560 | 0.360 | 146 | 35.5 | 3.53 | 49.1 | 12.2 | 2.04 |
| W6 \( \times 25 \) | 7.34 | 6.38 | 6.080 | 0.455 | 0.320 | 53.4 | 16.7 | 2.70 | 17.1 | 5.61 | 1.52 |
| W5 \( \times 19 \) | 5.54 | 5.15 | 5.030 | 0.430 | 0.270 | 26.2 | 10.2 | 2.17 | 9.13 | 3.63 | 1.28 |
| 16 | 4.68 | 5.01 | 5.000 | 0.360 | 0.240 | 21.3 | 8.51 | 2.13 | 7.51 | 3.00 | 1.27 |
| W4 \( \times 13 \) | 3.83 | 4.16 | 4.060 | 0.345 | 0.280 | 11.3 | 5.46 | 1.72 | 3.86 | 1.90 | 1.00 |

† A wide-flange shape is designated by the letter W followed by the nominal depth in inches and the weight in pounds per foot.

\[ \text{Price per ton} = \$15 \text{ mile} \]
Appendix C. Properties of Rolled-Steel Shapes
(U.S. Customary Units)

W Shapes
(Wide-Flange Shapes)

\[ S = \frac{I}{c} = \frac{I}{d/2} \]

<table>
<thead>
<tr>
<th>Designation</th>
<th>Area, ( A, \text{in}^2 )</th>
<th>Depth, ( d, \text{in.} )</th>
<th>Width, ( b, \text{in.} )</th>
<th>Thickness, ( t, \text{in.} )</th>
<th>Thickness, ( t, \text{in.} )</th>
<th>Axis X-X</th>
<th>Axis Y-Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>W10 × 200</td>
<td>22.3</td>
<td>16.65</td>
<td>2.44</td>
<td>0.215</td>
<td>20000</td>
<td>110</td>
<td>5.8</td>
</tr>
<tr>
<td>W12 × 200</td>
<td>28.9</td>
<td>16.85</td>
<td>2.60</td>
<td>0.235</td>
<td>20000</td>
<td>110</td>
<td>5.8</td>
</tr>
<tr>
<td>W14 × 200</td>
<td>35.2</td>
<td>17.24</td>
<td>2.75</td>
<td>0.255</td>
<td>20000</td>
<td>110</td>
<td>5.8</td>
</tr>
<tr>
<td>W16 × 200</td>
<td>42.6</td>
<td>17.72</td>
<td>2.90</td>
<td>0.275</td>
<td>20000</td>
<td>110</td>
<td>5.8</td>
</tr>
<tr>
<td>W18 × 200</td>
<td>50.0</td>
<td>18.24</td>
<td>3.06</td>
<td>0.295</td>
<td>20000</td>
<td>110</td>
<td>5.8</td>
</tr>
</tbody>
</table>

A wide-flange shape is designated by the letter W followed by the nominal depth in inches and the weight in pounds per foot.

(Table continued on page 752)
Design the lightest W to carry

\[ FOS = 1.6 \]
\[ A992 \text{ Steel} \]
Based on yield

\[ S = \frac{M}{\sigma} \]
\[ M = 150 \text{kft} \]
\[ S = \frac{(150 \text{kft})(12 \text{ in/ft})}{(50 \text{ k/ln}^2)} = 57.6 \text{ in}^3 \]

SHAPE

- W10 x 54
- W12 x 50
- W14 x 43
- W16 x 40
- W18 x 35
- W21 x 44
- W24 x 68
- W27 x 84
- W30 x 99
- W33 x 118

Use

- W18 x 35
- W21 x 44

Pg A18

\[ (64.7) \]
\[ (62.7) \]

Pg A12

\[ 60.9 \text{ in}^3 \]
\[ 64.2 \]
\[ 62.6 \]
\[ 64.7 \]
\[ 57.6 \]
\[ 81.6 \]
Design lightest rod to carry $60k$ at $20'$

$\text{Yall} = 40 \text{ksi}$

**Solution:**

Length is not important

$\sigma = \frac{P}{A}$ so $A = \frac{P}{Yall} = \frac{60k}{40 \text{ kpsi}} = 1.5 \text{in}^2$

$A = \frac{\pi D^2}{4} = 1.5 \text{in}^2$

$D = \sqrt{\frac{1.5 \text{in}^2 (\frac{4}{\pi})}{\frac{4}{3\pi}}} = 1.38''$
defined as the reciprocal of the radius of curvature $\rho$, and can be obtained by solving Eq. (4.9) for $1/\rho$:

$$\frac{1}{\rho} = \frac{\epsilon_m}{c} \quad \text{(4.20)}$$

But, in the elastic range, we have $\epsilon_m = \sigma_m/E$. Substituting for $\epsilon_m$ into (4.20), and recalling (4.15), we write

$$\frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{E} \frac{Mo}{I}$$

or

$$\frac{1}{\rho} = \frac{M}{EI}$$

(4.21)