UNIAXIAL STRESS-STRAIN

Stress-Strain Curve for Mild Steel

The slope of the linear portion of the curve equals the modulus of elasticity.

DEFINITIONS

Engineering Strain

\[ \varepsilon = \frac{\Delta L}{L_o}, \text{ where} \]

\( \varepsilon \) = engineering strain (units per unit),
\( \Delta L \) = change in length (units) of member,
\( L_o \) = original length (units) of member.

Percent Elongation

\[ \% \text{ Elongation} = \left( \frac{\Delta L}{L_o} \right) \times 100 \]

Percent Reduction in Area (RA)

The \( \% \) reduction in area from initial area, \( A_i \), to final area, \( A_f \), is:

\[ \% RA = \left( \frac{A_i - A_f}{A_i} \right) \times 100 \]

Shear Stress-Strain

\[ \gamma = \tau / G, \text{ where} \]

\( \gamma \) = shear strain,
\( \tau \) = shear stress, and
\( G \) = shear modulus (constant in linear torsion-rotation relationship).

\[ G = \frac{E}{2(1 + \nu)} \text{, where} \]

\( E \) = modulus of elasticity (Young’s modulus)
\( \nu \) = Poisson’s ratio, and
= – (lateral strain)/(longitudinal strain).

Uniaxial Loading and Deformation

\[ \sigma = \frac{P}{A}, \text{ where} \]

\( \sigma \) = stress on the cross section,
\( P \) = loading, and
\( A \) = cross-sectional area.

\[ \varepsilon = \frac{\delta}{L}, \text{ where} \]

\( \varepsilon \) = elastic longitudinal deformation and
\( L \) = length of member.

\[ E = \sigma / \varepsilon = \frac{P}{A} \frac{L}{\delta} \]

\[ \delta = \frac{PL}{AE} \]

True stress is load divided by actual cross-sectional area whereas engineering stress is load divided by the initial area.

THERMAL DEFORMATIONS

\[ \delta_t = \alpha L (T - T_o), \text{ where} \]

\( \delta_t \) = deformation caused by a change in temperature,
\( \alpha \) = temperature coefficient of expansion,
\( L \) = length of member,
\( T \) = final temperature, and
\( T_o \) = initial temperature.

CYLINDRICAL PRESSURE VESSEL

Cylindrical Pressure Vessel

For internal pressure only, the stresses at the inside wall are:

\[ \sigma_r = P_i \frac{r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_i \]

For external pressure only, the stresses at the outside wall are:

\[ \sigma_r = -P_o \frac{r_o^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_e, \quad \text{where} \]

\( \sigma_r \) = tangential (hoop) stress,
\( \sigma_r \) = radial stress,
\( P_i \) = internal pressure,
\( P_o \) = external pressure,
\( r_i \) = inside radius, and
\( r_o \) = outside radius.

For vessels with end caps, the axial stress is:

\[ \sigma_a = P_i \frac{r_i^2}{r_o^2 - r_i^2} \]

\( \sigma_a \), \( \sigma_r \), and \( \sigma_a \) are principal stresses.

When the thickness of the cylinder wall is about one-tenth or less of inside radius, the cylinder can be considered as thin-walled. In which case, the internal pressure is resisted by the hoop stress and the axial stress.

\[ \sigma_s = \frac{Pr}{t} \quad \text{and} \quad \sigma_a = \frac{Pr}{2t} \]

where \( t \) = wall thickness.

**STRESS AND STRAIN**

**Principal Stresses**

For the special case of a two-dimensional stress state, the equations for principal stress reduce to

\[ \sigma_{\text{max}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

\[ \sigma_x = 0 \]

\[ \sum \sigma = \tau_{xy} \]

The two nonzero values calculated from this equation are temporarily labeled \( \sigma_x \) and \( \sigma_y \), and the third value \( \sigma_3 \), is always zero in this case. Depending on their values, the three roots are then labeled according to the convention:
- algebraically largest = \( \sigma_1 \), algebraically smallest = \( \sigma_3 \),
- other = \( \sigma_2 \).

A typical 2D stress element is shown below with all indicated components shown in their positive sense.

**Mohr's Circle – Stress, 2D**

To construct a Mohr's circle, the following sign conventions are used.

1. Tensile normal stress components are plotted on the horizontal axis and are considered positive. Compressive normal stress components are negative.
2. For constructing Mohr's circle only, shearing stresses are plotted above the normal stress axis when the pair of shearing stresses, acting on opposite and parallel faces of an element, forms a clockwise couple. Shearing stresses are plotted below the normal axis when the shear stresses form a counterclockwise couple.

The circle drawn with the center on the normal stress (horizontal) axis with center, \( C \), and radius, \( R \), where

\[ C = \frac{\sigma_x + \sigma_y}{2}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

The two nonzero principal stresses are then:

\[ \sigma_x = C + R \]
\[ \sigma_y = C - R \]

The maximum inplane shear stress is \( \tau_{\text{in}} = R \). However, the maximum shear stress considering three dimensions is always

\[ \tau_{\text{max}} = \frac{\sigma_x - \sigma_3}{2}. \]

**Hooke's Law**

Three-dimensional case:

\[ \epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \]
\[ \epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \]
\[ \epsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \]

Plane stress case (\( \sigma_z = 0 \)):

\[ \epsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y) \]
\[ \epsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x) \]
\[ \epsilon_z = - \frac{1}{E}(\nu \sigma_x + \sigma_y) \]

Uniaxial case (\( \sigma_z = 0 \)):

\[ \sigma_x = E \epsilon_x \text{ or } \sigma = E \epsilon, \text{ where} \]
\( \sigma_x, \sigma_y, \sigma_z \) = normal stress,
\( \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \) = shear stress,
\( \tau_{xy}, \tau_{yz}, \tau_{zx} \) = shear stress,
\( E \) = modulus of elasticity,
\( G \) = shear modulus, and
\( v \) = Poisson’s ratio.

---

\( p = 200 \text{ psi} \)
\( t_y = 36 \text{ ksi} \)
\( FOS = 4 \)

\[ \sigma_H = \frac{pR}{t} = \frac{36 \text{ ksi}}{4} \text{ allowed} = \frac{t_y}{4} \]

\[ t_{\text{reqd}} = \frac{pR}{t_{\text{all}}} = \frac{(200 \text{ kips/ft}^2)(1.5 \text{ ft})(12')}{(36 \text{ kips/ft}^2 \times 1000 \text{ kips})/4} \]

\[ = 0.4 \text{ in} \]
\[ t_y = 36 \text{ ksi}/4 = 9 \text{ ksi} \]

\[ \sigma_x = \frac{t_{\text{reqd}}}{2} = 4.5 \text{ ksi} \]

\[ \sigma_y = \frac{t_x + t_y}{2} + \frac{(t_x - t_y)^2}{2} \]

\[ \sigma_x = \frac{t_x + t_y}{2} + \frac{(t_x - t_y)^2}{2} + t_{xy} \]

\[ \sigma_{xy}' = \frac{t_x + t_y}{2} \]

\( \theta = 27^\circ \)

\[ T_{\text{max}} = \sqrt{\left( \frac{t_y^2/4 - 36/4}{2} \right)^2 + 0^2} \]
\[ T_{\text{max \ in \ plane}} = \pm \frac{18/4 - 36/4}{2} = 2.25 \text{ ksi} \]

\[ T_{\text{max \ out \ of \ plane}} = \frac{16/4}{2} = 2.25 \text{ ksi} \]

\[ W_{\text{INS}} = 4.5 \text{ ksi} \]

\[ T_{\text{max \ out \ of \ plane}} = \frac{18/4}{2} = 2.25 \text{ ksi} \]

\[ p_o = 14.7 \text{ psi} \]

\[ p = 2.147 \text{ psi} \]
MOHR'S CIRCLE

\[ \frac{36}{4} = \sigma_y - \sigma_p \]

\[ \frac{18}{4} = \sigma_x = \sigma_p \]

\[ V\left(\frac{18}{4}, 0\right) \]

\[ H\left(\frac{36}{4}, 0\right) - 0 \]

Max out of plane = 4.5

Max in plane = 2.25

IN PLANE STRESSES
Beams with unsymmetric loadings and overhanging beams will be

9.2. DEFORMATION OF A BEAM UNDER TRANSVERSE LOADING

\[ \frac{1}{\rho} = \frac{M(x)}{EI} \quad \text{Eq. 9.2} \]

Consider, for example, a cantilever beam \( AB \) of length \( L \) subjected to a concentrated load \( P \) at its free end \( A \) (Fig. 9.3a). We have \( M(x) = -Px \) and, substituting into (9.1),

\[ \frac{1}{\rho} = \frac{Px}{EI} \]

which shows that the curvature of the neutral surface varies linearly

\[ \sum M_{\text{ext}} = 0 = M_{\text{max}} = Px \]
9.3. EQUATION OF THE ELASTIC CURVE

We first recall from elementary calculus that the curvature of a plane curve at a point \( Q(x,y) \) of the curve can be expressed as

\[
\Gamma = \frac{d^2 y}{dx^2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}
\]

where \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \) are the first and second derivatives of the function \( y(x) \) represented by that curve. But, in the case of the elastic curve of a beam, the slope \( \frac{dy}{dx} \) is very small, and its square is negligible compared to unity. We write, therefore,

\[
\frac{1}{\rho} = \frac{d^2 y}{dx^2}
\]

(9.3)

Substituting for \( 1/\rho \) from (9.3) into (9.1), we have

\[
\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}
\]

(9.4)

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

†It should be noted that, in this chapter, \( y \) represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

\[
\frac{1}{\rho} = \frac{M}{EI} \quad \text{Eq. 4.21}
\]

\[
EI \frac{d^2 y}{dx^2} = M(x)
\]
The Curvature of Any Curve

The curvature $K$ of a curve at $P$ is the limit of its average curvature for the arc $PQ$ as $Q$ approaches $P$. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}$$

Curvature in Rectangular Coordinates

$$K = \frac{y'}{\left[1 + (y')^2\right]^{3/2}}$$

When it may be easier to differentiate the function with respect to $y$ rather than $x$, the notation $x'$ will be used for the derivative.

$$x' = \frac{dx}{dy}$$

$$K = \frac{-x'}{\left[1 + (x')^2\right]^{3/2}}$$

The Radius of Curvature

The radius of curvature $R$ at any point on a curve is defined as the absolute value of the reciprocal of the curvature $K$ at that point.

$$R = \frac{1}{|K|} \quad (K \neq 0)$$

$$R = \frac{\left[1 + (y')^2\right]^{3/2}}{|y'|} \quad (y' \neq 0)$$

$$\frac{1}{R} = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1 + (\frac{dy}{dx})^2\right]^{3/2}}$$
The product $EI$ is known as the **flexural rigidity** and if it varies

\[
\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}
\]

where $C_1$ is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent to the elastic curve at $Q$ forms with the horizontal (Fig. 9.7), and recalling that this angle is very small, we have

\[
\frac{dy}{dx} = \tan \theta = \theta(x)
\]

Thus, we write Eq. (9.5) in the alternative form

\[
EI \theta(x) = \int_0^x M(x) \, dx + C_1
\]

Integrating both members of Eq. (9.5) in $x$, we have

\[
EI y = \int_0^x \left[ \int_0^x M(x) \, dx + C_1 \right] dx + C_2
\]

\[
EI y = \int_0^x dx \int_0^x M(x) \, dx + C_1 x + C_2
\]

Fig. 9.8 Boundary conditions for statically determinate beams.
The cantilever beam AB is of uniform cross section and carries a load P at its free end A (Fig. 9.9). Determine the equation of the elastic curve and the deflection and slope at A.

\[ \sum M_{\text{cut}} = 0 = +Px + M_x \]

![Fig. 9.9](image)

Using the free-body diagram of the portion AC of the beam (Fig. 9.10), where C is located at a distance x from end A, we find

\[ M_x = -Px \]

Substituting for \( M_x \) into Eq. (9.4) and multiplying both members by the constant \( EI \), we write

\[ EI \frac{d^2y}{dx^2} = -Px \]

Integrating in x, we obtain

\[ EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + C_1 \]

We now observe that at the fixed end B we have \( x = L \) and \( \theta = \frac{dy}{dx} = 0 \) (Fig. 9.11). Substituting these values into (9.8) and solving for \( C_1 \), we have

\[ C_1 = \frac{1}{2}PL^2 \]

which we carry back into (9.8):

\[ EI \frac{dy}{dx} = -\frac{1}{2}Px^2 + \frac{1}{2}PL^2 \]

Integrating both members of Eq. (9.9), we write

\[ EI y = -\frac{1}{3}Px^3 + \frac{1}{3}PL^2x + C_2 \]

But, at B we have \( x = L, y = 0 \). Substituting into (9.10), we have

\[ 0 = -\frac{1}{3}PL^3 + \frac{1}{3}PL^3 + C_2 \]

\[ C_2 = \frac{1}{3}PL^3 \]

Carrying the value of \( C_2 \) back into Eq. (9.10), we obtain the equation of the elastic curve:

\[ EI y = -\frac{1}{3}Px^3 + \frac{1}{3}PL^2x - \frac{1}{3}PL^3 \]

or

\[ y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \]

The deflection and slope at A are obtained by letting \( x = 0 \) in Eqs. (9.11) and (9.9). We find

\[ y_A = \frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left( \frac{dy}{dx} \right)_A = \frac{PL^2}{2EI} \]

![Fig. 9.11](image)
\[ EI \theta_x = \left\{ M_x \, dx \right\} = \left\{ -\frac{wx^2}{2} \right\} \]

\[ EI \theta_x = -\frac{wx^3}{2 \cdot 3} + C_1 \]

B.C. (@x = L): \( \theta = 0 = -\frac{wL^3}{2 \cdot 3} + C_1 \)

\[ C_1 = +\frac{wL^3}{2 \cdot 3} \]

\[ EI \theta_x = -\frac{wx^3}{2 \cdot 3} + \frac{wL^3}{2 \cdot 3} \]
\[ EI \frac{y}{Q} = \int \left( -\frac{wx^3}{2 \cdot 3} + \frac{WL^3}{2 \cdot 3} \right) dx \]

\[ EI \frac{y}{Q} = -\frac{wx}{2 \cdot 3 \cdot 4} + \frac{WL^3}{2 \cdot 3} + C_2 \]

B.C.) \[ EI \frac{y}{Q} \bigg|_{x=L} = 0 = -\frac{WL^4}{2 \cdot 3 \cdot 4} + \frac{WL^3L}{2 \cdot 3} + C_2 \]

\[ C_2 = +\frac{WL^4}{2 \cdot 3 \cdot 4} - \frac{WL^4}{2 \cdot 3} = -\frac{WL^4}{8} \]

\[ EI \frac{y}{Q} = -\frac{wx^4}{2 \cdot 3 \cdot 4} + \frac{WL^3x}{2 \cdot 3} - \frac{WL^4}{8} \]

Max slope = \[ -\frac{wL^3}{2 \cdot 3} + \frac{WL^3}{2 \cdot 3} = \frac{WL^3}{6EI} \]

Max deflection = \[ -Q + 0 - \frac{WL^4}{8EI} \]
\[ \Sigma M_{cut} = 0 = P \left( \frac{x}{3} \right) + M \]

\[ M = - \frac{P L}{3} = - \frac{w x^2 L}{24} \]

\[ M = - \frac{w x^3}{6L} \]

\[ \frac{w}{L} = \frac{q_f}{x} \]

\[ q_f = \frac{w x}{L} \]

\[ P = \frac{1}{2} (x_c) w \]
\[ M = M_{\text{applied}} \]

\[ EI \frac{d^2y}{dx^2} = M_x = -M_{\text{applied}} = -M_0 \]

\[ \sum M_{\text{cut}} = 0 = +M_0 + M_x \]

\[ M_x = -M_{\text{applied}} = -M_0 \]
\[ EI \frac{d^4y}{dx^4} = EI \Theta_x = \int -M_0 dx \]

\[ = -M_0 x + C_1 \]

**B.C. @ x = L:** \( \Theta_L = 0 = -M_0 L + C_1 \)

So \( C_1 = M_0 L \)

So \( EI \Theta_x = -M_0 x + M_0 L \)

\[ EI \frac{d^2y}{dx^2} = \int (-M_0 x + M_0 L) dx \]

\[ = -\frac{M_0 x^2}{2} + M_0 L x + C_2 \]

**B.C. @ x = L:** \( y = 0 = -\frac{M_0 L^2}{2} + M_0 L^2 + C_2 \)

So \( C_2 = \frac{M_0 L^2}{2} - M_0 L^2 = -\frac{M_0 L^2}{2} \)

So \( EI \frac{d^2y}{dx^2} = -\frac{M_0 x^2}{2} + M_0 L x - \frac{M_0 L^2}{2} \)

\[ W \rightarrow E = 30 \times 10^6 \text{ psi} \]

\[ I = 117 \text{ in}^4 \]

\[ L = 20 \text{ ft} \]

\[ M = 26 \text{ kft} \]

\[ \frac{I}{E} = 117 \text{ in}^4 \]

\[ I = 20 \text{ in}^2 \]

\[ \Theta_{max} = 0 = -\Theta + \frac{(26 \text{ kft})(12''/ft)(20\text{ ft}*12''/ft)}{30 \times 10^3 \text{ k}\text{in}^2/(117 \text{ in}^4)^2} \]

\[ = 2.56'' \quad \text{no good} \]

\[ If \quad S_{max} = \frac{L}{360} = \frac{20\text{ ft}*12''/ft}{360} = 0.67'' \quad \text{no good} \]
In a beam, the deflection and the slope of the beam are continuous at any point.

For the prismatic beam and the loading shown (Fig. 9.16), determine the slope and deflection at point D.

We must divide the beam into two portions, AD and DB, and determine the function $y(x)$ which defines the elastic curve for each of these portions.

1. From A to D ($x < L/4$). We draw the free-body diagram of a portion of beam AE of length $x < L/4$ (Fig. 9.17). Taking moments about $E$, we have

$$M_1 = \frac{3P}{4}x$$

(9.17)

or, recalling Eq. (9.4),

$$EI \frac{d^2 y_1}{dx^2} = \frac{3}{4}Px$$

(9.18)

where $y_1(x)$ is the function which defines the elastic curve for portion AD of the beam. Integrating in $x$, we write

$$EI \frac{dy_1}{dx} = \frac{3}{8}Px^2 + C_1$$

(9.19)

$$EI y_1 = \frac{1}{8}Px^3 + C_1x + C_2$$

(9.20)

2. From D to B ($x > L/4$). We now draw the free-body diagram of a portion of beam AE of length $x > L/4$ (Fig. 9.18) and write

$$M_2 = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right)$$

(9.21)

$$= -\frac{Px}{4} + \frac{PL}{4}$$

Fig. 9.16

Fig. 9.17

Fig. 9.18
or, recalling Eq. (9.4) and rearranging terms,

$$EI \frac{d^2y_2}{dx^2} = -\frac{1}{4}Px + \frac{1}{4}PL$$  \hspace{1cm} (9.22)

where $y_2(x)$ is the function which defines the elastic curve for portion $DB$ of the beam. Integrating in $x$, we write

$$EI \theta_2 = EI \frac{dy_2}{dx} = -\frac{1}{8}Px^2 + \frac{1}{4}PLx + C_3$$  \hspace{1cm} (9.23)

$$EI y_2 = -\frac{1}{24}Px^3 + \frac{1}{8}PLx^2 + C_3x + C_4$$  \hspace{1cm} (9.24)

**Determination of the Constants of Integration.** The conditions that must be satisfied by the constants of integration have been summarized in Fig. 9.19. At the support $A$, where the deflection is defined by Eq. (9.20), we must have $x = 0$ and $y_1 = 0$. At the support $B$, where the deflection is defined by Eq. (9.24), we must have $x = L$ and $y_2 = 0$. Also, the fact that there can be no sudden change in deflection or in slope at point $D$ requires that $y_1 = y_2$ and $\theta_1 = \theta_2$ when $x = L/4$. We have therefore:

$$[x = 0, y_1 = 0], \text{Eq. (9.20): } 0 = C_2$$  \hspace{1cm} (9.25)

$$[x = L, y_2 = 0], \text{Eq. (9.24): } 0 = \frac{1}{12}PL^2 + C_3L + C_4$$  \hspace{1cm} (9.26)

$$[x = L/4, \theta_1 = \theta_2], \text{Eqs. (9.19) and (9.23): }$$

$$\frac{3}{128}PL^2 + C_1 = \frac{7}{128}PL^2 + C_3$$  \hspace{1cm} (9.27)

$$[x = L/4, y_1 = y_2], \text{Eqs. (9.20) and (9.24): }$$

$$\frac{PL^3}{512} + C_1 \frac{L}{4} = \frac{11PL^3}{1536} + C_3 \frac{L}{4} + C_4$$  \hspace{1cm} (9.28)

Solving these equations simultaneously, we find

$$C_1 = -\frac{7PL^2}{128}, \hspace{1cm} C_2 = 0, \hspace{1cm} C_3 = -\frac{11PL^2}{1536}, \hspace{1cm} C_4 = \frac{PL^3}{384}$$

Substituting for $C_1$ and $C_2$ into Eqs. (9.19) and (9.20), we write that for $x \leq L/4$,

$$EI \theta_1 = \frac{3}{8}Px^2 - \frac{7PL^2}{128}$$  \hspace{1cm} (9.29)

$$EI y_1 = \frac{1}{8}Px^3 - \frac{7PL^2}{128}x$$  \hspace{1cm} (9.30)

Letting $x = L/4$ in each of these equations, we find that the slope and deflection at point $D$ are, respectively,

$$\theta_D = -\frac{PL^2}{32EI} \hspace{1cm} \text{and} \hspace{1cm} y_D = -\frac{3PL^3}{256EI}$$

We note that, since $\theta_D \neq 0$, the deflection at $D$ is not the maximum deflection of the beam.
1) \( \Sigma F_v = 0 \)
2) \( \Sigma F_H = 0 \)
3) \( \Sigma M = 0 \)

\[ y = \frac{wL^2}{8} \]