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Network design for a grid hybrid transit service

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Abstract

In this paper, we develop an analytical model that aids decision-makers in designing a hybrid grid network that integrates a flexible demand responsive service with a fixed route service. The objective of the model is to determine the optimal number of zones in an area where each zone is served by a number of on-demand vehicles. The function of the on-demand vehicles is to transfer passengers to a fixed route line if the destination is to a different zone or to its final destination if it is within the same zone.

1. Introduction

The passage of the American with Disabilities Act (ADA) has changed the landscape for demand responsive transit (DRT) systems. First, the demand for this type of transit service has experienced tremendous growth (Levine, 1997). As a result, DRT ridership has nearly doubled and operating costs have increased to \$1.2 billion annually, more than 6% of the national budget for public transportation in 2000 (Federal Transit Administration 2000). Second, besides the increase in demand, ADA also set strict guidelines for the providers on trip denials and on-time performance (Lewis et al., 1998). In essence, transit agencies today are expected to provide better services while experiencing increased usage for demand responsive transit systems.

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These systems are highly subsidized. The National Transit Summaries and Trends (NTST) report for 2000, the most recent year available, indicates that the average cost per passenger trip for DRT systems is $16.74 with fares ranging from $1.50–3.00. By way of contrast, the NTST report indicates that the average cost per trip for fixed route lines is $2.19 with fares being roughly 25% of the cost. With this high of a subsidy for DRT systems and the continued growth, the operation of these systems which is not optional for transit services as mandated by ADA, will put a tremendous strain on the budget of transit agencies. We aim to address this important problem by studying innovative methods that can reduce the cost for this type of service. In particular, we study a grid hybrid transit service that integrates two modes of transportation: demand responsive service and fixed route service. In this type of service, local service is provided by on-demand vehicles and line-haul service is provided by a grid fixed route transit system, thus, minimizing the amount of travel required by the on-demand vehicles.

There has been some work in developing operational scheduling and routing policies for hybrid systems. Liaw et al. (1996) develop a scheduling heuristic based on a system in Ann Arbor, Michigan. Hickman and Blume (2000) develop an insertion heuristic and test it on a data set from Houston, Texas. Aldaihani and Dessouky (2003) develop a tabu search heuristic and test it on a data set from Antelope Valley in California. They show that shifting some of the demand to a hybrid service route (18.6% of the requests) reduces the on-demand vehicle distance by 16.6% without significantly increasing the trip times.

The above research studied a hybrid service from an operational point of view by developing algorithms to improve the scheduling of such a system. In this paper, we develop a model to aid decision-makers in designing a hybrid network. We study a problem where the service area is divided into zones with grid fixed route service. Each zone is served by a number of on-demand vehicles, which transfer passengers to a fixed route line if the destination is to a different zone or to its final destination if it is within the same zone. Our model determines the number of zones, the number of on-demand vehicles, the number of fixed route lines, and the number of fixed buses in each route that minimize the total cost which is a function of both operator and passenger cost.

The transit network design problem has been well studied by a number of researchers (see e.g., Newell, 1979; Mandl, 1980; Ceder and Wilson, 1986; LeBlanc, 1988; Chang and Schonfeld, 1991a; Baaj and Mahmassani, 1991, 1995; Chien and Schonfeld, 1997; Shrivastav and Dhingra, 2001). In these problems, the objective is to minimize some function that combines operator and passenger cost. The operator cost is usually represented as a function of the number of vehicles and miles traveled by the vehicles while the passenger cost is a function of their trip travel times.

There also have been studies that compare the performance of a fixed service with that of flexile service using solely on-demand vehicles. Adebisi and Hurdle (1982) develop a model that determines which service is more cost-effective depending on the ridership. Clearly, under low ridership the flexible service is cheaper while with high ridership the fixed route service is the more preferred method. Their study focuses on a single line service. Chang and Schonfeld (1991b) perform the comparison of feeder bus services. They show that a flexible subscription service is advantageous with smaller service areas and as a function of a number of other parameters. Jacobson (1980) compares a strictly flexible service (i.e., door-to-door) with a hybrid service (i.e., flexible feeder with fixed route service). The comparison is based on using the analytical models of Daganzo (1978) and Daganzo et al. (1977). This analysis shows that flexible service is less costly. However, his analysis assumes a very low demand density (which is not reflective of the current demand for such systems) and each zone (subarea) is serviced by a single vehicle. Our approach differs from this work by allowing for more than one on-demand vehicle in each zone. Furthermore, the ride time on the fixed service in Jacobson's model assumes direct service from the center of one zone to another. Since our model assumes a grid structure, the ride time on the fixed route line is a function of the number of zones. These differences necessitate the development of a different analytical model from the previously developed models to determine the number of required zones in a hybrid service. We note that a simulation analysis can be used to determine the number of zones. However, this requires building a simulation model of the system. Our purpose is to develop simple analytical equations to guide planners in determining the number of zones.

2. Network design

We assume that transportation requests (trips) arrive in a square (service area) at a constant rate λ. The trips' origin and destination locations are uniformly distributed within the square. Origin and destination locations are assumed to be independent. The square (service area) is partitioned into n^2 zones, each a square of side $1/n$. The system consists of two components: demand responsive service for pickup/delivery within each zone, and fixed route service for travel between zones. We assume that the number of on-demand vehicles in each zone is the same. The purpose for the on-demand vehicles is to move passengers within the zone. It serves one passenger at a time, directly from the pickup point to drop-off point, without intermediate stops and no ridesharing. The fixed route service follows a grid structure, as shown in Fig. 1. The purpose of the fixed route service is to transfer passengers traveling between zones. We define another decision variable m to represent the number of buses per fixed route. The objective is to develop an analytical model to determine the optimal number of service zones, n^2.

Note that if $n = 1$, there will be only one zone with no need for a fixed route service. As n increases, passengers are more likely to require a fixed bus line to travel to their final destination. Clearly, from a passenger perspective, they would prefer fewer zones, thereby minimizing the need for a transfer. However, since typically the cost per passenger mile to an agency of an on-demand trip is much higher than that of a fixed route trip, a transit agency from a cost perspective would prefer more zones. Our analytical model trades-off these two factors to determine the optimal number of zones.

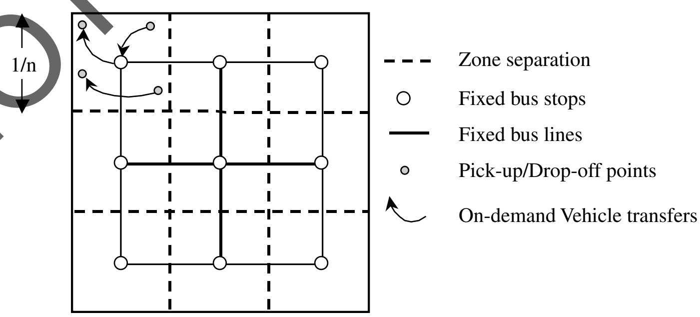

Fig. 1. Hybrid network ($n = 3$).

The decision variables, n and m, which need to be optimized in this problem, affects the network in different ways. For example, by increasing n, we will have:

1. More and smaller zones, which leads to more bus stops.
2. More fixed bus lines, which lead to more buses (number of fixed bus lines $= 2nm$).
3. Shorter distance between bus stops (distance $= 1/n$).
4. Shorter average traveling distance from origin points to entry bus stops and from exit bus stops to destination points.
5. Smaller likelihood that the pickup and delivery points for a trip are located in the same zone.

On the other hand, by increasing m, we will clearly have reduced average passenger waiting time at the bus stop, but also increased fixed costs.

3. Model

We next describe the model that determines the optimal number of zones, n^2, and the number of buses on each fixed route, m.

3.1. Assumptions

The assumptions of the model are as follows:

1. Maximum number of transfers for a passenger during an entire trip is three. That is, a passenger can only ride at most two on-demand vehicles and two buses to reach his/her final destination.
2. The network is symmetric and the distance between any two points in each zone is the Euclidean distance.
3. Ridesharing on an on-demand vehicle is not considered in the model, and there is no waiting costs associated with on-demand vehicles. We note that with a high demand rate the waiting for on-demand vehicles may not be negligible.
4. The number of on-demand vehicles in each zone is the same.
5. The trips are classified into three types:
   a. *Type 1*. Origin and destination are located in the same zone. These trips are served using strictly the demand responsive (curb-to-curb) service. The probability of having this type of trip is P_1.
   b. *Type 2*. Origin and destination are located in two zones that have a common fixed bus line. These trips can be satisfied by exactly one fixed bus line and two different on-demand vehicles, one at each zone. Also, for this type, the passenger waits for the fixed route bus only once, which is at the entry bus stop. The probability of having this type of trip is P_2.
   c. *Type 3*. Origin and destination are located in two zones that do not have a common fixed bus line. These trips can be satisfied by exactly two fixed bus lines and two on-demand vehicles. Moreover, this kind of trip has two alternative paths. In this type, the passenger waits for the fixed route bus twice, first at the entry bus stop and second at the connection bus stop. The probability of having this type of trip is P_3.

6. Based on assumption 5, the passenger can be in three states: in an on-demand vehicle, waiting at a bus stop for a fixed route bus, or in a fixed route bus. We assume the different states of a passenger may have a different cost to a passenger. That is, the inconvenience to a passenger for waiting at a stop for a bus may be higher than traveling on a bus.
7. The fixed line bus speed is independent of the demand. Fewer buses would mean more passengers per bus. The assumption is that the time between stops will not be affected by the demand. In practice, this time depends on the demand due to the dependence on the number of boardings and disembarkments. However, for model simplification we assume independence.
8. The on-demand vehicle is assumed to wait at the last delivery spot until the next demand arrives.

3.2. Parameters and notation

The following are the parameters of the model.

λ	arrival rate (passengers/day)
L	edge's length of the square service area (miles)
a_v	passenger cost of traveling in an on-demand vehicle ($/passenger/min)
a_b	passenger cost of traveling in a fixed route bus ($/passenger/min)
a_w	passenger cost of waiting at a bus stop ($/passenger/min)
V_v	variable operating cost for an on-demand vehicle ($/vehicle/min)
F_v	fixed cost of an on-demand vehicle ($/vehicle/day)
T_b	total cost (fixed + variable) of a fixed route bus ($/bus/day)
s_v	average speed of an on-demand vehicle (miles/min)
s_b	average speed of a fixed route bus (miles/min)
μ	maximum distance that can be traveled per on-demand vehicle per day (miles)
t_v	the time required to board and disembark an on-demand vehicle (min)
t_b	the time required to board and disembark a fixed route bus (min)

Note that, for the buses, T_b includes both fixed and variable costs; while, for the on-demand vehicle, they are separated: F_v represents the fixed costs and V_v the variable costs.

The computed variables in the model, that are a function of n and m, are:

dv_1	average distance traveled by a passenger on an on-demand vehicle between any pickup/delivery point in the zone and the bus stop
dv_2	average distance traveled by a passenger on an on-demand vehicle between any pickup and delivery point within a zone (for type 1 request)
de_1	average distance traveled by an on-demand vehicle empty from a delivery point to the bus stop in the zone
de_2	average distance traveled by an on-demand vehicle empty from a delivery point to the next pickup point within the zone
db	average distance traveled by a passenger on a fixed route bus (on only one fixed bus line)
wb	average passenger waiting time at the bus stop
δ	expected total daily distance traveled by on-demand vehicles in each zone
η	number of on-demand vehicles required per zone

Note that the above variables (except η) are computed in miles and refer to a square mile service area. We will need to multiply them by L to fit our model. Although the given parameters are based on a per day basis, we do not imply a uniform rate throughout the day. The system design should be based on a peak time demand.

3.3. Average on-demand vehicle distance traveled per passenger

In computing the total miles traveled by an on-demand vehicle within a zone, we assume that after a drop-off, a vehicle moves directly to pickup another passenger. Depending on the passenger type, the average on-demand vehicle distance traveled per customer is determined as follows:

- For passenger type 1 the distance is $\mathrm{de}_2 + \mathrm{dv}_2$; one trip empty to pickup the passenger (de_2) and one trip with the passenger to drop him/her off (dv_2).
- For passenger types 2 and 3 the distance is $\mathrm{de}_2 + 2\mathrm{dv}_1 + \mathrm{de}_1$; in the pickup zone one trip empty to pickup the passenger (de_2) and one trip with the passenger to drop him/her off at the fixed bus stop (dv_1), in the drop-off zone one trip empty to pickup the passenger at the fixed bus stop (de_1) and one trip with the passenger to drop him/her off at destination (dv_1).

Fig. 2 illustrates how the expected on-demand vehicle travel distance is computed on a hybrid network consisting of 4 zones ($n = 2$). In this figure, we show two different requests. Passenger type 1 needs to travel only within zone II. Passenger type 2 needs to travel from zone I to zone III. The dots mark the current location of the on-demand vehicles.

3.4. Total Cost function definition

The total cost of designing the network consists of three major components: passenger cost, on-demand vehicle cost and fixed route bus cost. The passenger cost has only a variable cost, which depends on the trip time. However, each state of a passenger has a different variable cost. The on-

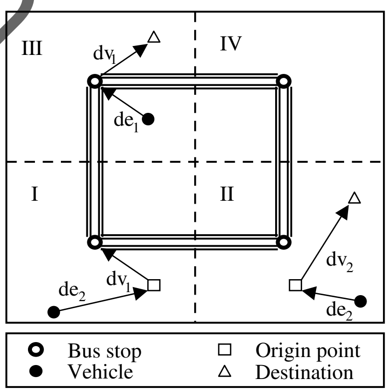

Fig. 2. Expected vehicle travel distance.

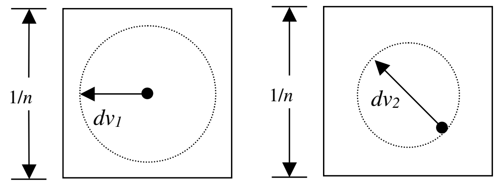

Fig. 3. Average distances dv_1 and dv_2.

demand vehicle cost has both a variable and fixed cost component. The bus cost has only a fixed cost component. Thus:

$$\text{Total Cost} = \text{Passengers Cost} + \text{On-demand Vehicle Cost} + \text{Bus Cost}$$

$$\begin{cases} \text{Passengers Cost} & = \text{Type 1 Cost} + \text{Type 2 Cost} + \text{Type 3 Cost} \\ \text{On-demand Vehicle Cost} & = \text{Fixed Cost} + \text{Variable Cost} \\ \text{Bus Cost} & = \text{Total bus Cost} \end{cases}$$

$$\begin{aligned} \text{Total Cost}^2 = {} & P_1\lambda(L\mathrm{dv}_2/s_\mathrm{v} + t_\mathrm{v})a_\mathrm{v} + P_2\lambda[2(L\mathrm{dv}_1/s_\mathrm{v} + t_\mathrm{v})a_\mathrm{v} + L(\mathrm{wb})a_\mathrm{w} + (L\mathrm{db}/s_\mathrm{b} + t_\mathrm{b})a_\mathrm{b}] \\ & + P_3\lambda[2(L\mathrm{dv}_1/s_\mathrm{v} + t_\mathrm{v})a_\mathrm{v} + L(2\mathrm{wb})a_\mathrm{w} + 2(L\mathrm{db}/s_\mathrm{b} + t_\mathrm{b})a_\mathrm{b}] + F_\mathrm{v}(\eta)Ln^2 \\ & + V_\mathrm{v}\lambda[(1-P_1)(L(\mathrm{de}_2 + 2\mathrm{dv}_1 + \mathrm{de}_1)/s_\mathrm{v} + 2t_\mathrm{v}) + P_1(L(\mathrm{de}_2 + \mathrm{dv}_2)/s_\mathrm{v} + t_\mathrm{v})] \\ & + T_\mathrm{b}(2n)m \end{aligned} \tag{1}$$

4. Derivation of the terms used in the Total Cost function

In this section, we derive the terms used in the Total Cost function in terms of n and m. These terms are dv_1, dv_2, de_1, de_2, P_1, P_2, P_3, db and wb.

4.1. Average on-demand vehicle distance traveled by customers (dv_1 and dv_2)

In each zone, we would like to find the expected distances dv_1 and dv_2 as a function of n. dv_1 is the expected distance from the bus stop, which represents the center of the zone, to any other point in the zone, while dv_2 is the expected distance between any two points in the zone (Fig. 3).

For uniformly distributed points in a unit square area, Larson and Odoni (1981) showed that:

$$\mathrm{dv}_1 = 0.383 \ * \ (1/n) \tag{2}$$

$$\mathrm{dv}_2 = 0.5214 \ * \ (1/n) \tag{3}$$

4.2. Average distance traveled empty by the on-demand vehicle (de_1 and de_2)

Upon completion of a trip, the on-demand vehicle must travel to a new pickup to begin its next assignment. This travel distance depends on the dispatching strategy and traffic level. If vehicles

[2] The last term of the expression $[T_b(2n)m]$ should be zero when $n=1$, because no buses are needed.

Table 1
Average per passenger on-demand vehicle distance

Demand assumption	Average on-demand vehicle distance	
	Passenger type 1 ($de_2 + dv_2$)	Passenger types 2 and 3 ($de_2 + 2dv_1 + de_1$)
Low ($de_1 = dv_1$; $de_2 = dv_2$)	$2dv_2$	$3dv_1 + dv_2$
High ($de_1 = de_2 = 0$)	dv_2	$2dv_1$

are dispatched on a first-come-first-serve basis, as might be expected under light traffic, then de_1 and de_2 are well approximated by Eqs. (2) and (3). Under heavy traffic, the queue of requests awaiting service could become long, in which case vehicles might be dispatched to the nearest available pickup request, making de_1 and de_2 close to zero. Table 1 summarizes the average on-demand vehicle distance traveled per passenger for the two above cases. In reality, the average distance traveled will be between these two cases depending on the level of demand. In the experimental section, we analyze the sensitivity of the results to the assumption of low and high demand.

4.3. Probability of the three passenger types P_1, P_2, and P_3

In this section, we find the probability of each type of trip in terms of n. The probability of each passenger type equals its number of outcomes divided by the total number of outcomes. Next, the number of outcomes is determined with respect to the value of n.

1. Outcome = trip (origin zone, destination zone) (e.g. (2,3) of a trip means that the origin point is in zone 2 and the destination point is in zone 3).
2. Number of zones = n^2.
3. Total number of outcomes = n^4.
4. Total number of type 1 trip outcomes (both endpoints are in the same zone) = n^2.
5. Total number of type 2 trip outcomes (endpoints are located in two zones that have a common fixed bus line) = $2(n-1)n^2$.
6. Total number of type 3 trip outcomes (endpoints are located in two zones that do not have a common fixed bus line) = $(n^2 - 2(n-1) - 1)n^2$.

$$P_1 = \frac{n^2}{n^4} = \frac{1}{n^2} \tag{4}$$

$$P_2 = \frac{2(n-1)n^2}{n^4} = \frac{2(n-1)}{n^2} \tag{5}$$

$$P_3 = \frac{(n^2 - 2n + 1)n^2}{n^4} = \frac{(n-1)^2}{n^2} \tag{6}$$

$$\sum_{i=1}^{3} P_i = \frac{1}{n^2} + \frac{2(n-1)}{n^2} + \frac{(n-1)^2}{n^2} = 1 \tag{7}$$

4.4. Average bus distance traveled by customers (db)

Below we find the average distance traveled by the passengers using the bus on one fixed bus route. This average distance is for passenger types 2 and 3. Note that passenger type 1 does not use buses, passenger type 2 uses one fixed bus line (db), and passenger type 3 uses two fixed bus lines (2db). The tables below show the methodology for calculating the average distance for different values of n. Table 2 classifies the bus trips for each value of n according to the number of bus stops that the passenger may visit and counts the number for each category.

For example, when $n = 3$, there are only two types of bus trips, which are "1" when the passenger visits only one bus stop and "2" when the passenger visits exactly two bus stops. Also, note that there are 4 possible outcomes of the "1" type and 2 possible outcomes of the "2" type. To illustrate the idea in more detail, the set of all the possible outcomes of the "2" type when n equals 3 is {(1,2), (2,1), (2,3),(3,2), (1,3), (3,1)}. Note that outcome (1,2) of a trip means that the pickup point is located in zone 1 and the delivery point is located in zone 2. Let:

N_n all possible outcomes of type 2 requests on a fixed bus line with n zones
D_n cumulative distance for all possible outcomes of type 2 requests on a fixed bus line with n zones
db_n expected distance traveled by type 2 passengers on the fixed route bus with n zones

$$N_n = \sum_{i=1}^{n-1} 2i \tag{8}$$

$$D_n = \frac{1}{n}\sum_{i=1}^{n-1} 2i(n-i) \tag{9}$$

$$\mathrm{db}_n = \frac{D_n}{N_n} = \frac{\frac{1}{n}\sum_{i=1}^{n-1} 2i(n-i)}{\sum_{i=1}^{n-1} 2i} \tag{10}$$

Table 2
Number of bus trips

# of visited stops	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
1	2	4	6	8	10	12	14	16	18
2		2	4	6	8	10	12	14	16
3			2	4	6	8	10	12	14
4				2	4	6	8	10	12
5					2	4	6	8	10
6						2	4	6	8
7							2	4	6
8								2	4
9									2
N_n	2	6	12	20	30	42	56	72	90

In the previous step, the objective is to only count the number of each type of bus trip. These trips are distinct from each other in terms of the distance. In Table 3, the number of each bus trip type is converted into distance by applying Eq. (9). By applying Eq. (10), Table 4 shows the average distance traveled by the passenger using the fixed route buses for different values of n.

For the purpose of simplicity, we set the average distance db_n to its converging value of 0.34 (Fig. 4). Note that $db_{100} \approx db_{1000} \approx 0.34$.

Table 3
Length of bus trips

# of visited stops	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
1	1	1.333	1.5	1.6	1.667	1.714	1.75	1.778	1.8
2		1.333	2.0	2.4	2.667	2.857	3.00	3.111	3.2
3			1.5	2.4	3.000	3.429	3.75	4.000	4.2
4				1.6	2.667	3.429	4.00	4.444	4.8
5					1.667	2.857	3.75	4.444	5.0
6						1.714	3.00	4.000	4.8
7							1.75	3.111	4.2
8								1.778	3.2
9									1.8
D_n	1	2.667	5.0	8.0	11.667	16.000	21.00	26.667	33.0

Table 4
Average bus trip

# of visited stops	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
D_n	1	2.667	5.000	8.0	1.667	16.000	21.000	26.667	33.000
N_n	2	6.000	12.000	20.0	30.000	42.000	56.000	72.000	90.000
db_n	0.5	0.444	0.417	0.4	0.389	0.381	0.375	0.370	0.367

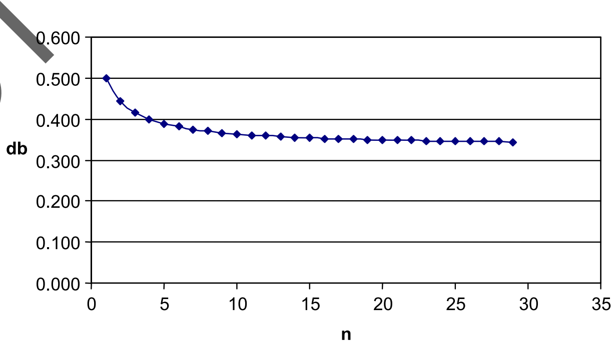

Fig. 4. db as a function of n.

4.5. Average waiting time at the bus stop (wb)

Osuna and Newell (1972) show that the expected waiting time of passengers at a bus stop, wb, who arrive at random times and are independent of the bus schedule is equal to:

$$\mathrm{wb} = H/2 \ * \ (1 + C_\mathrm{v}^2) \tag{11A}$$

where H is the mean headway (service interval) and C_v is the coefficient of variation of the headways. Assuming that C_v is zero and each fixed route has only one bus, the expected passenger waiting time can be represented by one half of the service interval at each bus stop. A bus arrives at each bus stop every $2(n-1)/n$ unit distance. This is the time it takes to make a round trip assuming that the stops are located in the center of the zone. Therefore, for a unit square service area ($L = 1$):

$$\mathrm{wb} = (n-1)/(n * \ s_\mathrm{b}) \tag{11B}$$

However, if we have more than one bus in the fixed line, an arbitrary length L of the square service area and the time between these buses are equal, the average waiting time at the bus stop is given by Eq. (11B) divided by the number of buses in the fixed line (m), multiplied by L;

$$\mathrm{wb} = L * \ (n-1)/(n * \ m * \ s_\mathrm{b}) \tag{11C}$$

If the fixed bus schedules are appropriately synchronized, the waiting time at transfer points between two fixed lines (for passenger type 3) could be close to zero; but assuming unsynchronized timing, wb is a correct estimate for those cases too.

4.6. Fixed cost of on-demand vehicles

To find the fixed cost of the on-demand vehicles, we need to compute the expected total daily on-demand vehicle travel miles in each zone, δ, which depends on the arrival rate and the probability of the three passenger types. Eq. (12) states that the vehicles fixed cost equals the cost of having an on-demand vehicle times the number of on-demand vehicles per zone times the number of zones. Eq. (13) states that the number of vehicles per zone, η, equals the expected daily on-demand vehicle travel miles in each zone divided by the maximum number of miles that can be traveled by a vehicle.

$$\text{Vehicles Fixed Cost} = F_\mathrm{v}\eta n^2 \tag{12}$$

$$\eta = \lceil \delta/\mu \rceil \tag{13}$$

Although η is a discrete variable, we will assume it is a continuous variable in order to get a continuous function for n. Below we determine the expected daily on-demand vehicle travel miles in each zone:

$$\begin{aligned}
\delta &= \frac{\lambda P_1(\mathrm{de}_2 + \mathrm{dv}_2)}{n^2} + \frac{\lambda P_2(\mathrm{de}_2 + 2\mathrm{dv}_1 + \mathrm{de}_1)}{n^2} + \frac{\lambda P_3(\mathrm{de}_2 + 2\mathrm{dv}_1 + \mathrm{de}_1)}{n^2} \\
&= \frac{\lambda P_1(\mathrm{de}_2 + \mathrm{dv}_2)}{n^2} + \frac{\lambda(1 - P_1)(\mathrm{de}_2 + 2\mathrm{dv}_1 + \mathrm{de}_1)}{n^2} \\
&= \frac{\lambda(\mathrm{de}_2 + 2\mathrm{dv}_1 + \mathrm{de}_1) + \lambda(1/n^2)(\mathrm{dv}_2 - 2\mathrm{dv}_1 - \mathrm{de}_1)}{n^2}
\end{aligned} \tag{14a}$$

Therefore:

$$\text{For low demand } (\mathrm{de}_1 = \mathrm{dv}_1, \mathrm{de}_2 = \mathrm{dv}_2), \quad \delta = \frac{1.6704(\lambda/n) - 0.6276(\lambda/n^3)}{n^2} \tag{14b}$$

$$\text{For high demand } (\mathrm{de}_1 = \mathrm{de}_2 = 0), \quad \delta = \frac{0.766(\lambda/n) - 0.2446(\lambda/n^3)}{n^2} \tag{14c}$$

4.7. Total Cost function

In order to formulate the Total Cost function we assume low demand density ($\mathrm{de}_1 = \mathrm{dv}_1$, $\mathrm{de}_2 = \mathrm{dv}_2$). A sensitivity analysis will be performed at the end of the paper to observe the effect of high demand density assumption ($\mathrm{de}_1 = 0$, $\mathrm{de}_2 = 0$) over the optimality.

After finding all the terms needed in terms of n and m, the Total Cost function (Eq. (1)) can be stated as follows:

$$\begin{aligned}
\text{Total Cost} = f(n,m) &= \frac{1}{n^2}\lambda\left(0.5214\frac{L}{ns_\mathrm{v}} + t_\mathrm{v}\right)a_\mathrm{v} \\
&+ \left[\frac{2(n-1)}{n^2}\right]\lambda\left[2\left(0.383\frac{L}{ns_\mathrm{v}} + t_\mathrm{v}\right)a_\mathrm{v} + \left(\frac{n-1}{nms_\mathrm{b}}\right)La_\mathrm{w} + \left(0.34\frac{L}{s_\mathrm{b}} + t_\mathrm{b}\right)a_\mathrm{b}\right] \\
&+ \left[\frac{(n-1)^2}{n^2}\right]\lambda\left[2\left(0.383\frac{L}{ns_\mathrm{v}} + t_\mathrm{v}\right)a_\mathrm{v} + 2\left(\frac{n-1}{nms_\mathrm{b}}\right)La_\mathrm{w} + 2\left(0.34\frac{L}{s_\mathrm{b}} + t_\mathrm{b}\right)a_\mathrm{b}\right] \\
&+ F_\mathrm{v}\left[\frac{1.6704(\lambda/n) - 0.6276(\lambda/n^3)}{n^2\mu}\right]Ln^2 \\
&+ V_\mathrm{v}\lambda\left[\left(1 - \frac{1}{n^2}\right)\left(1.6704\frac{L}{ns_\mathrm{v}} + 2t_\mathrm{v}\right) + \frac{1}{n^2}\left(1.0428\frac{L}{ns_\mathrm{v}} + t_\mathrm{v}\right)\right] + T_\mathrm{b}(2n)m
\end{aligned} \tag{15}$$

Rearranging and ordering the terms of the equation, the function becomes:

$$\begin{aligned}
\text{Total Cost} &= f(n,m) \\
&= \frac{1}{n^3}\lambda L\left[-0.2446\frac{a_\mathrm{v}}{s_\mathrm{v}} - 0.6276\left(\frac{F_\mathrm{v}}{\mu} + \frac{V_\mathrm{v}}{s_\mathrm{v}}\right)\right] \\
&+ \frac{1}{n^2m}\left(2\frac{\lambda La_\mathrm{w}}{s_\mathrm{b}}\right) + \frac{1}{n^2}[-\lambda t_\mathrm{v}(a_\mathrm{v} + V_\mathrm{v})] + \frac{1}{nm}\left(-4\frac{\lambda La_\mathrm{w}}{s_\mathrm{b}}\right) \\
&+ \frac{1}{n}\lambda\left[0.766\frac{La_\mathrm{v}}{s_\mathrm{v}} - 2a_\mathrm{b}\left(0.34\frac{L}{s_\mathrm{b}} + t_\mathrm{b}\right) + 1.6704L\left(\frac{F_\mathrm{v}}{\mu} + \frac{V_\mathrm{v}}{s_\mathrm{v}}\right)\right] + \frac{1}{m}\left(2\frac{\lambda La_\mathrm{w}}{s_\mathrm{b}}\right) \\
&+ 2\lambda\left[a_\mathrm{b}\left(0.34\frac{L}{s_\mathrm{b}} + t_\mathrm{b}\right) + t_\mathrm{v}(a_\mathrm{v} + V_\mathrm{v})\right] + nm(2T_\mathrm{b})
\end{aligned} \tag{16}$$

We next show the parameter values for which the Total Cost function is convex. A function of two variables is convex if its Hessian matrix is positive definite. Also, for a convex function, the global minimum can be found by identifying the value of the decision variables that makes the gradient equal zero (unless the minimum is attained on the border of the variables' set, in our case $m = 1$ or $n = 1$).

Assuming n and m continuous, the gradient and the Hessian matrix are as follows:

$$\text{Gradient}: \quad \Delta f(n,m) = \begin{bmatrix} \dfrac{\partial f(n,m)}{\partial n} \\ \dfrac{\partial f(n,m)}{\partial m} \end{bmatrix} \tag{17}$$

$$\begin{aligned} \frac{\partial f(n,m)}{\partial n} = {} & \frac{1}{n^4}\lambda L\left[0.7338\frac{a_v}{s_v} + 1.8828\left(\frac{F_v}{\mu} + \frac{V_v}{s_v}\right)\right] + \frac{1}{n^3 m}\left(-4\frac{\lambda L a_w}{s_b}\right) + \frac{1}{n^3}[2\lambda t_v(a_v + V_v)] \\ & + \frac{1}{n^2 m}\left(4\frac{\lambda L a_w}{s_b}\right) + \frac{1}{n^2}\lambda\left[-0.766\frac{L a_v}{s_v} + 2a_b\left(0.34\frac{L}{s_b} + t_b\right) - 1.6704L\left(\frac{F_v}{\mu} + \frac{V_v}{s_v}\right)\right] \\ & + m(2T_b) \end{aligned} \tag{18}$$

$$\frac{\partial f(n,m)}{\partial m} = \frac{1}{m^2}\left(\frac{1}{n^2} - \frac{2}{n} + 1\right)\left(-2\frac{\lambda L a_w}{s_b}\right) + n(2T_b) \tag{19}$$

$$\text{Hessian matrix}: \quad \nabla^2 f(n,m) = \begin{bmatrix} a_H & b_H \\ b_H & c_H \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 f(n,m)}{\partial n^2} & \dfrac{\partial^2 f(n,m)}{\partial n \partial m} \\ \dfrac{\partial^2 f(n,m)}{\partial m \partial n} & \dfrac{\partial^2 f(n,m)}{\partial m^2} \end{bmatrix} \tag{20}$$

$$\begin{aligned} \frac{\partial^2 f(n,m)}{\partial n^2} = {} & \frac{1}{n^5}\lambda L\left[-2.9352\frac{a_v}{s_v} - 7.5312\left(\frac{F_v}{\mu} + \frac{V_v}{s_v}\right)\right] + \frac{1}{n^4 m}\left(12\frac{\lambda L a_w}{s_b}\right) \\ & + \frac{1}{n^4}[-6\lambda t_v(a_v + V_v)] + \frac{1}{n^3 m}\left(-8\frac{\lambda L a_w}{s_b}\right) \\ & + \frac{1}{n^3}\lambda\left[1.532\frac{L a_v}{s_v} - 4a_b\left(0.34\frac{L}{s_b} + t_b\right) + 3.3408L\left(\frac{F_v}{\mu} + \frac{V_v}{s_v}\right)\right] \end{aligned} \tag{21}$$

$$\frac{\partial^2 f(n,m)}{\partial m^2} = \frac{1}{m^3}\left(\frac{1}{n^2} - \frac{2}{n} + 1\right)\left(4\frac{\lambda L a_w}{s_b}\right) \tag{22}$$

$$\frac{\partial^2 f(n,m)}{\partial n \partial m} = \frac{\partial^2 f(n,m)}{\partial m \partial n} = \frac{1}{m^2}\left(\frac{1}{n^3} - \frac{1}{n^2}\right)\left(4\frac{\lambda L a_w}{s_b}\right) + 2T_b \tag{23}$$

We note that if the fractional optimal value of n is small that rounding should not be used and the total cost for the two integer neighborhood solutions should be compared and the smallest selected.

5. Computational experiment

In this section, we demonstrate the capabilities of the model in determining the optimal configuration for a given set of parameters. Two cases are analyzed:

1. a relatively small service area of 100 square miles ($L = 10$), with a total demand λ of 1000 customers/day, a relatively high bus total costs (T_b) and a small customers' waiting costs (a_v, a_b, a_w);
2. a large metropolitan area of 900 square miles ($L = 30$), with a total demand λ of 10,000 customers/day, a relatively low bus total costs (T_b) and higher customers' waiting costs (a_v, a_b, a_w).

For both cases we find the minimum of the Total Cost function with respect to m and n. In addition, for Case 1 only, we perform sensitivity analysis over a various set of parameters.

5.1. Case 1

Table 5 shows the base case values of the input parameters to the model.

In order to find the optimal n and m which minimize the Total Cost function, we first take the partial derivative with respect to m (Eq. (19)) and show that it is positive for all values of n and m (in the relevant set $n \geqslant 1$, $m \geqslant 1$). Rearranging the terms of the equation, it is simple to verify that:

$$\begin{aligned} \text{Eq.}(19) > 0 \iff \frac{\partial f(n,m)}{\partial m} &= \frac{1}{m^2}\left(\frac{1}{n^2} - \frac{2}{n} + 1\right)\left(-2\frac{\lambda L a_w}{s_b}\right) + n(2T_b) \\ > 0 \iff \frac{(n-1)^2}{n^3} &< \frac{T_b s_b}{\lambda L a_w} m^2 \end{aligned} \tag{24}$$

Let us define

$$g(n) = \frac{(n-1)^2}{n^3}, \quad C = \frac{T_b s_b}{\lambda L a_w} \tag{25}$$

The function $g(n)$ achieves its maximum for $n = 3$ (in the space $n \geqslant 1$), with a value of $g(3) = 0.148$. Therefore since $g(n) \leqslant \max[g(n)]$ and $C \leqslant Cm^2$ for any value of $m \geqslant 1$, Eq. (24) is verified for all n and m as long as:

$$C > 0.148 = \max[g(n)] \tag{26}$$

Table 5
Parameter values

Parameter	Value	Unit
λ	1000	Customers/day
L	10	Miles
a_v	0.1	\$/customer/Min
a_b	0.1	\$/customer/Min
a_w	0.1	\$/customer/Min
μ	120	Miles/vehicle
V_v	1.5	\$/vehicle/Min
F_v	15	\$/vehicle/day
T_b	1000	\$/bus/day
s_v	0.4	Miles/Min
s_b	0.25	Miles/Min
t_v	0.1	Min
t_b	0.1	Min

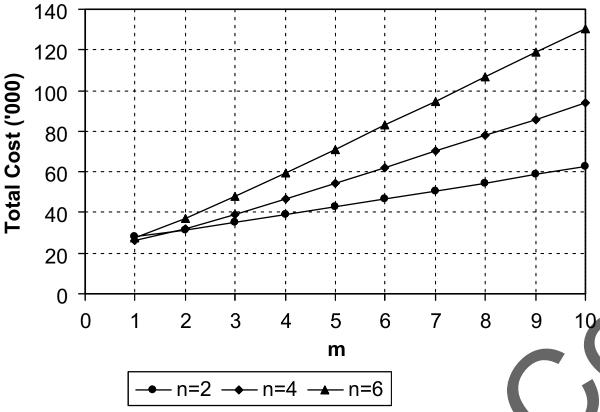

Fig. 5. Total Cost function vs. m.

The value of C depends on the input parameters. Assuming the above ones, we have $C = 0.25$, therefore Eq. (19) is always positive and we ascertain that the Total Cost function is monotonically increasing with m, for any value of n for this data set. Consequently the minimum of the Total Cost function, with respect to m, is reached for $m = 1$. Fig. 5 plots the Total Cost function vs. m, for three different values of n: as expected the function is monotonically increasing with m, confirming the above considerations.

We can now treat the Total Cost function as a function of only one variable, n, having determined that $m = 1$. Taking in consideration Eq. (21), that is the second derivative of the Total Cost function with respect to n, it is clear that it will be positive for all relevant values of n, in fact:

$$\frac{d^2 f(n)}{dn^2} = \frac{95,806}{n^3} + \frac{47,040}{n^4} - \frac{299,172}{n^5} > 0, \quad \forall n \geqslant 2 \tag{27}$$

Therefore, the Total Cost function is a convex function for these parameter values and any local minimum in the function is also a global minimum. Fig. 6 plots the Total Cost function vs. n. According to the figure, the minimum value [3] of n, among its integer values, is 5 (for $n = 1$, the convexity does not hold anymore, but it is easy to check that $f(1) > f(5)$). Hence, for this set of parameters a network design consisting of 25 service zones and 10 fixed bus lines is optimal.

Table 6 shows a breakdown of the components of the total cost as a function of n. As the table shows, as n increases there are less type 1 and 2 passengers and more type 3 passengers since it is more likely that a passenger will have origin and destination points located in different zones when there are many small zones. Also, the passenger cost and fixed bus cost increases with increasing number of zones since the passengers will have longer trip times due to transfers and there will be more fixed bus lines. However, the number of miles traveled by on-demand vehicles decreases. As this analysis shows, our model trades off these cost factors to determine the optimal number of zones.

[3] The actual minimum is reached for $n = 4.9$, but we need only consider integer values.

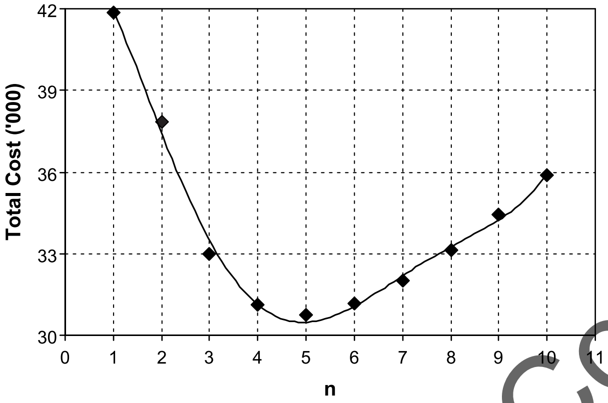

Fig. 6. Total Cost function vs. n (at $m = 1$).

Table 6
Total cost values

n	Probability			Cost components			Total cost
	P_1	P_2	P_3	Passenger	On-demand vehicle	Fixed bus	
1	1.000	0.000	0.000	1313	40,559	0	41,872
2	0.250	0.500	0.250	4268	29,587	4000	37,855
3	0.111	0.444	0.444	6017	20,958	6000	32,975
4	0.063	0.375	0.563	7044	16,092	8000	31,136
5	0.040	0.320	0.640	7710	13,045	10,000	30,755
6	0.028	0.278	0.694	8175	10,971	12,000	31,146
7	0.020	0.245	0.735	8518	9473	14,000	31,991
8	0.016	0.219	0.766	8781	8342	16,000	33,122
9	0.012	0.198	0.790	8988	7457	18,000	34,445
10	0.010	0.180	0.810	9157	6747	20,000	35,904

Table 7
Sensitivity analysis over λ and T_b

T_b	λ					
	250	500	1000	2000	3000	5000
250	5	7	10	11 ($m = 2$)	13 ($m = 2$)	17 ($m = 2$)
500	3	5	7	10	12	12 ($m = 2$)
1000	1	3	5	7	9	11
2000	1	1	3	5	6	8
3000	1	1	1	4	5	6

5.1.1. Sensitivity analysis

Table 7 shows the results of sensitivity analysis for the parameters λ and T_b on selecting the optimal number of zones. The other parameters described earlier remain unchanged. As the table

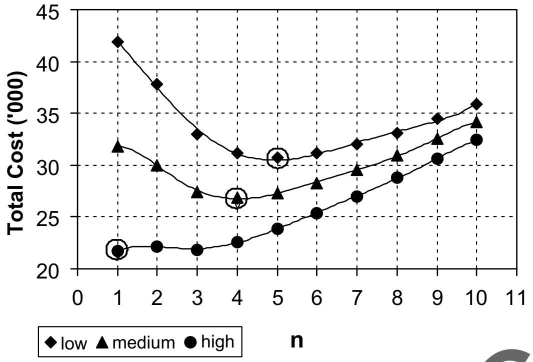

Fig. 7. Total Cost function vs. n, vs. demand density.

illustrates, the optimal number of zones increases as we increase λ and decrease T_b. Note that the sufficient condition $C > 0.148$ (Eq. (26)) needed to guarantee optimality at $m = 1$ does not hold for some of the cases, where optimal values are reached for $m = 2$.

We now consider how demand density affects the solution. The minimum value of the Total Cost function, with respect to the m variable, is still reached for $m = 1$, since Eq. (26) is not affected by the demand density assumption and the above considerations about the optimality over m still hold. On the other hand, the optimality over n is indeed influenced by the demand density. In fact, assuming high demand ($de_1 = de_2 = 0$), the optimality is attained for $n = 1$. Since the assumptions of case a and b in Section 4.2 describe two asymptotic cases regarding the values of de_1 and de_2, we can further reasonably assume that their actual estimates will be something in between those limit values, depending on the real demand density. Fig. 7 compares the Total Cost function curves assuming low demand ($de_1 = dv_1$; $de_2 = dv_2$), high demand ($de_1 = de_2 = 0$) and the middle point value ($de_1 = 0.5dv_1$; $de_2 = 0.5dv_2$). As we can observe the minimum for the middle point assumption is reached for $n = 4$, 16 service zones and 8 fixed bus routes.

5.2. Case 2

As a second experiment we analyze a case where the service area represents a large metropolitan region. Table 8 lists the parameters' values for this case.

Note that in this case Eq. (26) is not valid anymore, because $C = 0.0042$, therefore optimality is not necessarily attained at $m = 1$. This is not surprising, since we decreased the total cost for buses (T_b) and we increased the customers' waiting costs (other than increasing both L and λ), intuitively causing the optimal m that minimizes the cost to become larger.

The optimality conditions, in terms of n and m, are found by imposing the Gradient of the Total Cost function (Eq. (17)) equal to zero. Hence, setting Eq. (19) to zero and rearranging the terms, we are able to find m as a function of n:

$$m = \sqrt{\frac{\lambda L a_w}{T_b s_b} \frac{n-1}{n\sqrt{n}}} \tag{28}$$

Table 8
Parameter values

Parameter	Value	Unit
λ	10,000	Customers/day
L	30	Miles
a_v	1.5	\$/customer/Min
a_b	1.5	\$/customer/Min
a_w	1.5	\$/customer/Min
μ	120	Miles/vehicle
V_v	1.5	\$/vehicle/Min
F_v	15	\$/vehicle/day
T_b	500	\$/bus/day
s_v	0.4	Miles/Min
s_b	0.25	Miles/Min
t_v	0.1	Min
t_b	0.1	Min

Substituting this value of m in Eq. (18) and setting it equal to zero, a function of a single variable, n, is obtained:

$$\begin{aligned}&\frac{1}{n^4}\lambda L\left[0.7338\frac{a_v}{s_v}+1.8828\left(\frac{F_v}{\mu}+\frac{V_v}{s_v}\right)\right]+\frac{1}{n^3}[2\lambda t_v(a_v+V_v)]\\&\quad+\frac{1}{n^2}\lambda\left[-0.766\frac{La_v}{s_v}+2a_b\left(0.34\frac{L}{s_b}+t_b\right)-1.6704L\left(\frac{F_v}{\mu}+\frac{V_v}{s_v}\right)\right]\\&\quad+\left(\frac{1}{n\sqrt{n}}+\frac{1}{\sqrt{n}}\right)\left(2\sqrt{\frac{\lambda La_wT_b}{s_b}}\right)=0\end{aligned}\tag{29}$$

Note that the second, fourth and last terms of Eq. (18), which contain the m variable, merge together to generate the last term of Eq. (29).

With the above parameters, Eq. (29) is solved for $n = 24.1$ and, using Eq. (28), $m = 3.02$. Since the solution has to be in the integer domain, the minimum value of the Total Cost function is reached at $n = 24$ and $m = 3$, as shown also in Fig. 8, where the Total Cost function is plotted. We

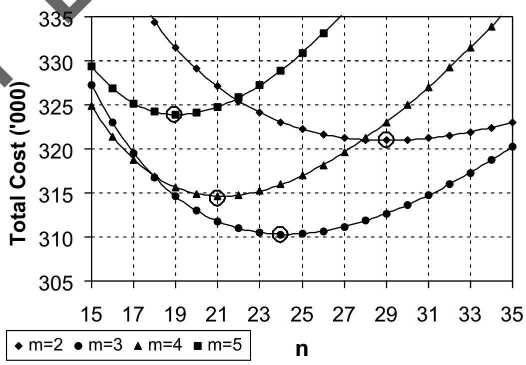

Fig. 8. Total Cost function vs. n and m.

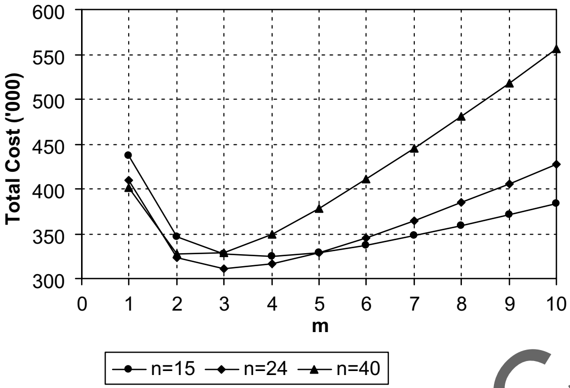

Fig. 9. Total Cost function vs. m at different n.

also note that this optimal solution yields a high headway and constraints can be added to restrict the search space if headway considerations need to be included in the model.

Hence, a network design consisting of 576 service zones and 48 fixed bus lines, and 3 buses for each line is optimal.

Note that the Total Cost function is monotonically increasing with m, for $m > 4$, for any n, as shown in Fig. 9.

6. Conclusions

An analytical model has been developed to aid decision-makers in designing a hybrid grid network that consists of two modes of transportation. The objective of the model is to determine the optimal number of zones in an area. Each zone is served by a number of on-demand vehicles, which transfer passengers to a fixed route line if the destination is in a different zone or to its final destination if it is within the same zone. Our model trades off the cost to passengers, on-demand vehicles, and fixed bus lines to determine the optimal number of zones.

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