GENERAL VOLUMETRIC CONSTITUTIVE EQUATION FOR GEO-MATERIALS

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ABSTRACT. A general volumetric constitutive equation for geomaterials is presented. It includes the pressure-volume and the time-volume behaviour under isotropic stresses as well as a change in permeability with volume in porous media with variable degree of saturation. A general equation has been applied to soils and to the compressibility behaviour of solids in water. Some other applications of it are suggested.

INTRODUCTION

General compressibility equation for soils has already been presented by the author (1981) using some general philosophic principles. The philosophic ideas have also been used to obtain a general permeability change equation for soils (1983a) and to obtain also the solution of volume changes when the isotropic stresses are instantaneously increased (1985). This paper presents a unified theory that contains all the above general theories, making emphasis in its philosophic foundation and suggesting some additional applications to mechanics of geo-materials.

PHILOSOPHICAL FOUNDATION

A. George Birkhoff, a mathematician from Harvard University (Tahan Malba, ) defines beauty in Fine Arts, Music, etc., by

\[ \text{Beauty} = \frac{\text{Order}}{\text{Complexity}} \]  

The author feels that nature is simple and beautiful and that eq. (1) applies also to science. Therefore, a true theory, should be simple and ordered. Order is achieved in a theory, through non linear "proper" proportionality.

Let us consider a physical phenomenon described by the "proper" variables \( x \) and \( y \) such that \( y = y(x) \) and \( x \) have a real domain from 0 to \( \infty \). (Definitions of "proper" variables and functions will not be given at this time, for example, when considering volume changes, the volume \( V \) as well as \((1+e)\) are proper variables, while the void ratio \( e \) is not). Let also the "proper" extreme values of \( y \) be \( y_0 = y(O) \) and \( y_\infty = y(\infty) \). Variable \( x \) has a complete real domain, from 0 to \( \infty \), while the real domain of variable \( y \) is not complete. Let \( z = z(y) \) be the corresponding "proper" function of \( y \) with complete real domain, that is, \( z(y_0) = 0 \) and \( z(y_\infty) = \infty \). A characteristic of this function \( z(y) \) is that it is the "simplest" function satisfying the conditions mentioned.

We may now enunciate the following two complementary basic philosopchic principles.

"The equation relating \( y \) and \( x \) may exist only through a non dimensional parameter and should, independently of critical points, satisfy the extreme boundary conditions, namely, \( y_0 = y(O) \) and \( y_\infty = y(\infty) \)."

"The relationship between \( y \) and \( x \) may exist only through the corresponding proper function \( z \) and should possess non linear proper proportionality."

The relationship between \( y \) and \( x \), should be given, therefore, by

\[ \frac{dz}{z} = a \frac{dx}{x} \]  

(2)

where \( a \) is a non dimensional parameter of proportionality. Eq. (2) defines "non linear proper proportionality" between \( y \) and \( x \) through the corresponding proper function \( z = z(y) \).

The proper function \( z = z(y) \) assuming that \( y_\infty < y_0 \) is simply given by

\[ z = \frac{1}{y - y_\infty} - \frac{1}{y_0 - y_\infty} \]  

(3)

Observe that in eq. (3), \( z(y_0) = 0 \) and \( z(y_\infty) = \infty \). If \( y_\infty > y_0 \) the reciprocals of \( y \) are to be used in eq. (3).

GENERAL EQUATION

Let \( y_1 = y(x_1) \) be a known value of \( y \) for \( x = x_1 (x_1 \neq 0) \) and \( z_1 = z(y_1) \) be the corresponding value of the proper function \( z \). Integrating eq. (2) between the limits \((x_1, z_1)\) and \((x, z)\) we get

\[ \ln \frac{z}{z_1} = a \ln \frac{x}{x_1} \]

and

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\[ \frac{z}{z_i} = \left(\frac{x}{x_i}\right)^a \quad (4) \]

Introducing eq. (3) into eq. (4) we may write
\[ \frac{y - y_\infty}{y - y_\infty} = 1 + \left(\frac{y - y_\infty}{y_i - y_\infty}\right) \left(\frac{x}{x_i}\right)^a \quad (5) \]

Eq. (5) may be written in a more simple form taking \( x_i \) as the value of \( x \) that makes \( y_i - y_\infty = 0.5 \). This value of \( x \) is called the "characteristic \( x^* \)". Therefore, we have
\[ \frac{y - y_\infty}{y - y_\infty} = 1 + \left(\frac{x}{x^*}\right)^a \quad (6) \]

Some applications of eqs. (3) to (6) follow. Practical applications are given in the references cited.

**VOLUME-PRESSURE EQUATION**

For the compressibility or expansibility of geo-materials, in the case of the first mechanical phase (before the critical point) (unvirgin) for solids (porous and non porous) and granular soils (rockfill, gravel and sand) me have (1981)
\[ \begin{align*}
  x &= \sigma, \quad y = V, \quad a = \gamma \\
  Y_0 &= V_0, \quad y_\infty = 0
\end{align*} \quad (7) \]
where \( \sigma \) is the isotropic pressure, \( V \) is the volume and \( \gamma \) is the non linear coefficient of compressibility.

Application of eqs. (3) to (6) give
\[ \frac{z}{V} = \frac{1}{V_0} \quad (8) \]
and
\[ \frac{1}{V} - \frac{1}{V_0} = \left(\frac{\sigma}{\sigma_i}\right)^\gamma \quad (9) \]

Eq. (9) may be written as eq. (5)
\[ \frac{V}{V_0} = 1 + \left(\frac{V}{V_i}\right)^{-1} \left(\frac{\sigma}{\sigma_i}\right)^\gamma \quad (10) \]
or in the simpler form, as eq. (6)
\[ \frac{V}{V_0} = 1 + \left(\frac{\sigma}{\sigma_i}\right)^\gamma \quad (11) \]

where \( \sigma^* \) is the "characteristic pressure", that is, the pressure that would reduce to half the original volume \( V_0 \) (If there were not before a critical point).

For the case of the second mechanical phase (virgin) of porous materials (like concrete) and granular soils as well as for plastic soils (plastic silts and clays) me have that \( V_0 = \infty \) and, therefore, \( z = \frac{1}{V} \) and eq. (9) reduces to
\[ \frac{V}{V_0} = \left(\frac{\sigma}{\sigma_i}\right)^\gamma \quad (12) \]

It has already been shown that eq. (12) applies also to the volumetric behaviour of plastic soils under triaxial conditions if \( \sigma \) are interpreted as the equivalent stresses (1969, 1975)

Eq. (12) applies to gases with \( \gamma = 1 \) for isothermal processes and \( \gamma < 1 \) for adiabatic processes.

**PERMEABILITY-VOLUME EQUATION**

For the variation of the Darcy's coefficient of permeability \( k \) with volume \( V \) for porous solids (soils included) me have (1983)
\[ \begin{align*}
  x &= v, \quad y = k, \quad a = \kappa \\
  Y_0 &= 0, \quad y_\infty = \infty
\end{align*} \quad (13) \]
where \( \kappa \) is the coefficient of permeability.

Application of eqs. (3) and (4) gives \( z = \kappa \) and
\[ \frac{k}{k_i} = \left(\frac{V}{V_i}\right)^{-\kappa} \quad (14) \]
or in the simpler form
\[ k = k_i (1 + e)^{-\kappa} \quad (15) \]
where \( k = k_i \) for \( e = 0 \).

Similarly for the variation of \( k \) with the degree of saturation \( S_r \), me have (1983)
\[ \begin{align*}
  x &= S_r, \quad y = k, \quad a = \kappa_s \\
  Y_0 &= 0, \quad y_\infty = \infty
\end{align*} \quad (16) \]

where \( \kappa_s \) is the coefficient of saturation permeability. Proceeding very similar than above me arrive at
\[ k = k_i (S_r)^{-\kappa_s} \quad (17) \]
where \( k = k_i \), for \( S_r = 1 \).

Equation (15) and (17) may be combined to res:
\[ k = k_i (1 + e)^{-\kappa}(S_r)^{-\kappa_s} \quad (18) \]

**TIME-VOLUME EQUATION**

For the evaluation of volume changes of geo-materials when the isotropic pressure is instantaneously increased (primary compression or consolidation of fine saturated soils excluded) me have (1983)
\[ \begin{align*}
  x &= t, \quad y = V, \quad a = \delta \\
  Y_0 &= V_0, \quad y_\infty = V_f
\end{align*} \quad (19) \]
where \( t \) is the time, \( V \) is the volume, \( V_0 \) and \( V_f \) are the initial and final volumes, and \( \delta \) is the coefficient of volume viscosity.

Application of eqs. (3) and (4) give
\[ \frac{z}{V} = \frac{1}{V - V_f} - \frac{1}{V_i - V_f} \quad (20) \]
and
\[
\frac{V_i - V_f}{V_i - V_f} - 1 = \left(\frac{V_i - V_f}{V_i - V_f} - 1\right) \left(\frac{t}{t_1}\right) \delta (21)
\]

the degree of compression \( U \) is defined by

\[
U = \frac{V_i - V}{V_i - V_f} (22)
\]

(21) may be written as

\[
\frac{1}{U} = 1 + \left(\frac{1}{U_f} - 1\right) \left(\frac{t}{t_1}\right) \delta (23)
\]

in the simpler form

\[
\frac{1}{U} = 1 + \left(\frac{t*}{t}\right) \delta (24)
\]

\( t* \) is the characteristic time, that is, the time needed to have \( U = 0.5 \).

\[ V_o = 1 + \left(\frac{1}{\gamma_u}\right) V \]

\[ V = \left(\frac{1}{\gamma}\right) V_o \]

\[ V = \left(\frac{1}{\gamma}\right) V_o \]

\[ \frac{\partial V}{\partial \sigma} = -\gamma \frac{V}{\sigma} (28) \]

Equating eqs. (27) and (28) at the point of tangency, where \( \sigma = \sigma^*, V = V^* \) we obtain

\[ \frac{V^*}{V_o} = 1 - \frac{\gamma}{\gamma_u} \]

and from eq. (26) we have

\[ \frac{\sigma^*}{1/\gamma} = \frac{V^*}{V_o} \]

This particular way of obtaining the critical point is only valid when the derivative is continuous at such a point. This appears to be so, at least for concrete, when the unvirgin curve is concave downwards (\( \gamma_u > 1 \)). A general way for obtaining the critical point is just equating eqs. (25) and (26) at such a point.

**COMPRESSIBILITY OF SUBSTANCES**

Fig. 2 shows the application of eq. (11) to various elements (Juarez-Badillo, E., 1984). The author directed a Master Thesis (Ortega-Rozo A., 1984) where eq. (11) was applied to more than 30 elements and more than 50 substances under isotropic pressures up to 100,000 kg/cm², using experimental data by P.W. Bridgman. The check was very good for the vast part of the substances. Small deviations appeared in some compounds and when the level of stress was indicating a polymorphic transition of the substance.

**THE CASE OF WATER**

Compressibility - Griner F.J. and Hewitt R. E. (1968) found for the relation between pressure and volume for water at 60°C for external pressures up to 12,000 atmospheres, using Bridgman's experimental data, the expression

\[ pV^6 = \text{constant} \]
where \( p = \sigma + i \), \( \sigma \) = external pressure, 
\( i \) = internal pressure = 3750 atmospheres.

Eq. (31) may be written in the form of eq. (12) as

\[
\frac{V}{V_o} = (\frac{\sigma + i}{\sigma_i + i})^{1/4}
\]  

(32)

and, therefore, for water \( \gamma = \frac{1}{5} \). It should be noted, however, the presence of the internal pressure. It would be interesting to know how it varies with temperature and if it is a characteristic of all liquids.

Temperature-Unit weight relation.- The unit weight of water \( \gamma_w \), at atmospheric pressure, as function of temperature \( T \) (Juarez-Badillo, E. and Rico-Rodriguez A., 1975), is described by eq. (4) if the point of maximum unit weight \( (T = 4^\circ C) \) is considered as a "zero" for the temperature. For this case we have

\[
x = T - 4^\circ C, \quad y = \gamma_w, \quad a = 1.75
\]

\[
Y_o = Y_w, \quad Y_o = 0
\]  

(33)

Application of eqs. (3) and (4) gives

\[
z = \frac{1}{Y_o} - \frac{1}{Y_w}
\]  

(34)

and

\[
Y_w = 1 + (\frac{Y_w}{Y_w} - 1)(\frac{T - 4^\circ C}{T})^1.75
\]

(35)

Using as known point \( T = 99^\circ C, \ Y_w = 0.9590 \) gr/cm\(^3\), we get

\[
\frac{1}{Y_w} = 1 + (\frac{1}{0.959} - 1)(\frac{T - 4^\circ C}{95})^{1.75}
\]

(36)

or in the simpler form

\[
\frac{1}{Y_w} = 1 + (\frac{T - 4^\circ C}{575.5})^{1.75}
\]

(37)

where use is made of the "characteristic temperature" \( T^* = 579.5^\circ C \), that is, the temperature that would make \( Y_w = 0.5 \) gr/cm\(^3\) if there were not a critical point at \( T = 100^\circ C \).

Eqs. (36) and (37) give \( Y_w \) with an approximation if \( 0.0001 \) gr/cm\(^3\). It would be interesting to know somewhat on the nature of the "zero" in the temperature variable and if it also presents in other liquids.

TEMPERATURE-VISCOSITY EQUATION FOR GASES

For the variation of the dynamic viscosity \( \mu \) and the kinematic viscosity \( \nu = \frac{\mu}{\rho} \) (where \( \rho \) is the specific mass) of gases at atmospheric pressure I have been using absolute temperature \( (^\circ K) \) for the temperature variable

\[
x = T^*K, \quad y = \mu, \quad a = a
\]

\[
y_o = 0, \quad y_\infty = \infty
\]  

(38)

Application of eqs. (3) and (4) gives

\[
\mu = \frac{\mu}{\rho}
\]

(39)

The variation of the specific mass \( \rho \) with temperature is simply given by \( (a = -1) \)

\[
\rho = \rho \left(\frac{T}{T_1}\right)^{-1}
\]

(40)

Dividing eq. (39) by eq. (40) we get

\[
\nu = \nu \left(\frac{T}{T_1}\right)^{-1+a}
\]

(41)

Eqs. (39) and (41) was applied to Helium (\( H_2 \)), Methane (\( CH_4 \)), Air, Oxygen (\( O_2 \)), Ammonia (\( NH_3 \)) and Carbon dioxide (\( CO_2 \)) (Roux H., 1949) with the result that, for the four gases \( a = 0.785 \), that is

\[
\nu = \nu \left(\frac{T}{T_1}\right)^{-1.785}
\]

(42)

while for the last two gases \( a = 1.05 \).

FINAL COMMENTS

The basic philosophical ideas enunciated above have been applied to the time-pressure-volume behaviour of geomaterials under isotropic stresses. Some limited application to the temperature-volume relation for water and temperature-viscosity relation for gases has been made. However, the author believes these are also applicable to other aspects of the mechanics of geo-materials, as for example, the deviatoric behaviour of geo-materials as also to partially saturated soils. If in the future this is found to be so, the philosophical principles enunciated may be referred to as "The Universal Principle of Scientific Beauty" and it would be the foundation of a new approach to the mechanics of geo-materials and may be referred to as Philosticity, due to its philosophical foundation.

CONCLUSIONS

The main conclusions are

1. Two complementary basic philosophical principles have been enunciated that are applicable to any physical phenomenon.

2. The application of the two basic physical principles conduces to the general constitutive equations (3) to (6).

3. Some general applications of eqs. (3) to (6) are to the volume-pressure equation, (11) and (12); permeability-saturation-volume equation (18); time-volume equation (24); water compressibility equation (32); etc. temperature-unit weight equation (37); gases temperature-viscosity equation (42).

4. The two complementary basic philosophical principles may well be the sound foundation for the new approach to the mechanics of geo-materials: Philosticity.

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