Sheet piles are vertical walls embedded in the ground to hold back water and/or soil. Figure 1 shows an example of a sheet pile used to separate two water bodies. Design of the sheet pile and understanding of the subsurface flow of groundwater around the pile and its foundation is a topic of geotechnical engineering. In this laboratory, we will explore a few properties of the soil and compute the groundwater flow using a finite difference solution to the governing differential equations.

Figure 1: Schematic of a sheet pile showing the dimensions of the pile and the surrounding soil.
Figure 2: Schematic of an infinitesimal soil element with the nine component of stress.

1 Principle Stresses in a Soil Sample

Stress is an important physical property in continuum mechanics, and expresses the force per unit area at a point. If we imagine an infinitesimal cubic element, as in Figure 2, the normal stresses are normal to each face of the cube, and the shear stresses are parallel to the cube faces. For a three-dimensional element, there are nine components of the stress, which combine to form a $3 \times 3$ tensor. In matrix form, the stress tensor is

$$\sigma = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}$$

(1)

where $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{zz}$ are the normal stresses in the $x$-, $y$-, and $z$-coordinate directions and $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$ are the shear stresses on their respective coordinate planes.

An important quantity in geotechnical design for a soil sample is the maximum normal stress occurring in the soil sample. This maximum stress occurs for an orientation in which all of the shear stress terms are zero; the corresponding stresses are the principle stresses, and the orientation is the principle axes of the element. A common means to find these stresses and their orientation is the graphical method called Mohr’s circle. A more direct mathematical method is to solve an eigenvalue problem.

We define the eigenvalue problem for the principles stresses $\sigma'$ as follows

$$|\sigma - \sigma'\mathbf{I}| = 0.$$

(2)

The corresponding eigenvectors $\phi_i$ are related to the orientation of the principle stresses. Since compression is positive in geotechnical engineering, these eigenvectors point in the opposite direction to the eigenvectors; hence, the principle axes directions are in the $-\phi_i$ directions.
1.1 Principle Stress Assignment

A soil sample from the site where we are to design a sheet pile wall has the following measured values

\[
\begin{align*}
\sigma_{xx} &= 14.0 \text{ MPa} \\
\sigma_{yy} &= 34.8 \text{ MPa} \\
\sigma_{zz} &= 16.1 \text{ MPa} \\
\tau_{xy} &= -0.6 \text{ MPa} \\
\tau_{yz} &= 6.0 \text{ MPa} \\
\tau_{xz} &= -2.1 \text{ MPa}
\end{align*}
\]  

1. Use the eigenvalue definition in Equation (2) to find the values of the principle stresses for this soil sample under this loading. Solve the eigenvalue problem by hand and then use your hand calculations to test the solution using the built-in Matlab function \texttt{eig}.

2. Solve for the eigenvectors that correspond to the principles stresses in the previous section. You should solve for at least one of the eigenvectors by hand and then compare your results to the solution from the Matlab \texttt{eig} function. These eigenvectors define a new coordinate system; hence, each vector should be orthogonal to the other vectors. Use dot products to show that the three eigenvectors you found are mutually orthogonal.

1.2 Memorandum and Questions

In your memorandum, report the values of your hand calculations and compare these to the solution from Matlab. Using the numerical solution, report the values of the principle stresses and corresponding eigenvectors, and show the Matlab output that verifies the eigenvectors are orthogonal. For results that are approximate (e.g., if a computation is 2.6475e-16 when it is supposed to be zero), discuss the types of error that are present in the numerical solution and whether the tests requested by the above steps are passing.

2 Groundwater Flow around a Sheet Pile

Because the soil underneath the two water bodies and surrounding the sheet pile in Figure 1 is silt, water can flow from the deeper water body on the left to the shallower water body on the right. The fundamental principle of groundwater fluid flow is Darcy’s law, which states that the flow rate \( \vec{q} \) is proportional to the hydraulic conductivity \( k \) and the local pressure gradient, evaluated from a quantity with dimensions of depth called the pressure head \( h \), namely

\[
\vec{q} = -k \nabla h.
\]  

The minus sign is necessary so that water flows from high to low pressure. If we consider the conservation of mass for every point in the domain, we can derive the two-dimensional, steady
groundwater flow equation for the \(xz\)-plane shown in Figure 1 given by

\[
\frac{\partial}{\partial x} \left[ k(x,z) \frac{\partial h}{\partial x} \right] + \frac{\partial}{\partial z} \left[ k(x,z) \frac{\partial h}{\partial z} \right] = S(x,z) \tag{5}
\]

where \(S(x,z)\) is a sink term (e.g., due to pumpage or recharge, which would be negative). At the soil-water interface between the standing water and the silt, the pressure head \(h\) is constant; thus, on those boundaries we have the boundary condition

\[
h(x,z) = h_b(x,z). \tag{6}
\]

where for this problem, \(h_b\) is equal to the water depth. At the bedrock on the bottom and along the sheet pile wall, we expect no groundwater flow; hence, \(q = 0\) on these boundaries. For the bottom boundary, we would have

\[
\frac{\partial h}{\partial z} = 0, \tag{7}
\]

and for the sheet pile, we would have

\[
\frac{\partial h}{\partial x} = 0. \tag{8}
\]

The two lateral boundaries within the soil matrix in reality are open boundaries, meaning that water can flow in or out depending on the conditions within and outside the domain we are evaluating. When solving Equation (5), however, we have to specify either a constant head, constant flux, or no flux condition on these lateral boundaries. The standard practice is to set a no-flux boundary on the lateral edges and to set these edges far enough from the sheet pile that this approximation is negligible. Hence, the boundary conditions on the sides will be

\[
\frac{\partial h}{\partial x} = 0. \tag{9}
\]

2.1 Numerical Solution

We will solve the groundwater flow equation and its boundary conditions using the finite difference method. Figure 3 shows the grid of nodes we will use to solve this problem. At each node in the grid, we must solve for the unknown pressure head, \(h\); hence, we will have \(n_x n_y\) equations in \(n_x n_y\) unknowns, where \(n_x\) is the number of grid cells in the \(x\)-direction and \(n_y\) is the number of grid cells in the \(y\)-direction.

We will organize our unknowns \(h_i\) in a single column vector. In Figure 3, the numbers written to the right of each node are the order we will number our unknowns. Node 1 is in the lower left corner, node numbers increase by 1 going up the first column, and the node numbers continue at the base of the next successive column, continuing across the numerical grid until we reach node \(n_x n_y\). Each node also has a two-dimensional coordinate \((x_i, y_j)\), which we could name using the integer values of \(i\) in the \(x\)-direction and \(j\) in the \(y\)-direction and a uniform grid size \(\Delta x = \Delta y\). These values are indicated by the labels on the left of and below the numerical domain. With this numbering system, an equation that computes the node number \(k\) given the \((i,j)\) index of a node
Figure 3: The numerical grid used to solve the groundwater flow problem showing the notation we use to label and number the nodes.

\[ k = (i - 1)n_y + j. \]  

(10)

We discretize the governing equation using a 5-point central difference. Figure 4 shows the finite difference stencil and the governing equation written for the five points at node \((i,j)\). If the conductivity is not constant, then it is common practice to use the geometric mean of the conductivity to approximate the value between each node, as shown in the figure. Thus, to fill the \(A\) matrix using the equations in the figure at each node, we would have the following for a point \((i,j)\) corresponding to node \(k\) inside the grid (e.g., not on a boundary):

\[
A(k,k-1:k+1) = \frac{[kb, -(kl + kr + kb + kt), kt]}{dx^2};
\]

\[
A(k,k-ny) = \frac{kl}{dx^2};
\]

\[
A(k,k+ny) = \frac{kr}{dx^2};
\]

and the corresponding value for the RHS vector \(b\) is

\[ b(k) = S(i,j) \]

In this laboratory, we will assume \(S\) is zero everywhere.

For the nodes at the top of the domain, the value of \(h\) is given from the boundary conditions. At these nodes, we have

\[
A(k,k) = 1
\]

\[
b(k) = h_b(i)
\]
Figure 4: The five-point central difference applied to the groundwater flow equation written for an internal node of the numerical domain.

For the nodes along the bottom no-flux boundary, the node value at the boundary should equal the node value one node above the boundary, hence

\[
A(k:k+1) = [1, -1];
\]

\[
b(k) = 0;
\]

For nodes on the left no-flux boundary, the corresponding evaluation is

\[
A(k, k) = 1;
\]

\[
A(k, k+ny) = -1;
\]

\[
b(k) = 0;
\]

And, for nodes on the right no-flux boundary, the correct expression is

\[
A(k, k) = 1;
\]

\[
A(k, k-ny) = -1;
\]

\[
b(k) = 0;
\]

For the sheet-pile wall, assume that the nodes along \(i = n_x/2\) are on the left-hand side of the boundary and the nodes along \(i = n_x/2 + 1\) are on the right. Use the appropriate no-flux boundary conditions modified from the expressions above for these nodes.

### 2.2 Groundwater Flow Assignment

For this problem, your program should solve the groundwater flow problem for the sheet pile and water levels in Figure 1 using 100 grid points in the \(x\)-direction and 50 grid points in the
y-direction using uniform spacing (e.g., $\Delta x = \Delta y$). To achieve this, start with a simple problem and gradually build-up complexity, following these steps.

For all of these problems, let the conductivity be selected from a normal distribution for silt given by

$$k_{\text{values}} = 1e-7 + 5e-8 \times \text{rand}(nx, ny);$$

1. Start by solving a simple problem using 6 points in the $x$-direction and 5 points in the $y$ direction. Your $A$ matrix will be $30 \times 30$; ignore the sheet pile wall. For my code, I have one loop to fill all internal nodes, separate loops for the left, right, and bottom boundaries, and a final loop for the constant head boundary. Try to use $n_x$ and $n_y$ instead of 6 and 5 throughout your code so that it is flexible to expand to larger grids. After you have filled the $A$ and $b$ matrices, the solution is

$$h = A \backslash b;$$

To plot the result, use

```matlab
% Build a square matrix of the solution vector
H = zeros(ny, nx)
for i = 1:nx
    for j = 1:ny
        % Let k be the number of the current node
        k = (i-1) * ny + j;
        H(j,i) = h(k);
    end
end
figure(1)
imagesc(flipud(H))
colorbar
axis equal
```

We have to flip the solution vector since Matlab plots the first row ($j = 1$) at the top of the figure.

2. Test the flexibility of your code by increasing the number of grid cells to 20 in the $x$-direction and 10 in the $y$-direction. Do not add in the sheet pile wall yet. If any parts of your code fail because you used a fixed value instead of a general expression that only depends on $n_x$ and $n_y$, etc., fix the problem by making the code more general. Test for other grid sizes until your code works properly for grid sizes larger and $6 \times 5$.

3. For the $20 \times 10$ grid, add the sheet pile wall at $i = 10$ to 11, and $j = 5$ to 10. Try not to hard-code these numbers, but rather use general expressions. It is ok to require the user to specify an even number of grid cells so that, for instance, $n_x/2$ will be an integer.
4. Finally, run your code for the full problem on the $100 \times 50$ grid. Also, test the case where a clay layer is applied for the left-hand side channel to a depth of 0.5 m with a constant conductivity of $k = 1 \cdot 10^{-11}$ m/s. Notice how the distribution of $h$ changes.

2.3 Memorandum and Questions

Include a section in your discussion of the groundwater solution that presents the results for the $6 \times 5$ sample domain. Discuss how you can validate that the numerical model is solving the equations correctly. For instance, are the values of $h$ on the top boundary equal to the boundary condition; are the values inside the domain more-or-less smooth (keeping in mind that this is a course grid); are the no-flux boundary conditions solved correctly. Include one other test in your memorandum of the model with more grid cells (e.g., $20 \times 10$) and without the sheet pile wall. Use this test to show that your code is general and able to solve for any number of grid cells.

Finally, solve the problem with the sheet pile for the $100 \times 50$ grid. What is the physical size of this numerical domain? Include separate figures for the sheet pile solution with and without the clay layer. You should plot the solution using `imagesc` as well as `contour`; be sure to label your axes. Compare the contour plot to Darcy’s law in Equation (4). Does the clay layer result in less seepage into the shallower channel from the deeper channel? How can you tell?

As always, include your model code in an appendix.