Introduction to Dispersion:

Why does the dye spread out much faster in the longitudinal direction than we expect? There is another advection effect we need to consider.

Note: if we used a 2D or 3D solution, we would have gotten the right answer with out accounting for dispersion.

Consider a channel flow:

\[ \text{dye injection stretches due to advection.} \]

\[ \text{later mixes due to diffusion.} \]

Shear flow + diffusion = dispersion.

Corresponding depth-averaged concentration profiles:

Dispersion: enhanced longitudinal spreading due to the combined effects of diffusion and velocity shear.
Modified Reynold's decomposition:

Turbulence:

\[ u(t) = \bar{u} + u'(t) \]

random turbulent fluctuation variable.

Dispersion

\[ u(z) = \bar{u} + u'(z) \]
deterministic (known) deviation variable.

\( u(z) \) is either a laminar profile or an average turbulent profile.

Taylor's analysis:

Substitute decomposition:

\[ u(z) = \bar{u} + u'(z) \]

\[ c(z) = \bar{c} + c'(z) \]

and take Depth Average.
Start with 3D turbulent diffusion Equation:
\[
\frac{dc}{dt} + \overline{u} \frac{dc}{dx} + \nabla \frac{dc}{dy} + \overline{w} \frac{dc}{dz} = D_{tx} \frac{\partial^2 c}{\partial x^2} + D_{ty} \frac{\partial^2 c}{\partial y^2} + D_{tz} \frac{\partial^2 c}{\partial z^2}
\]

Assume:
\[
\begin{align*}
\overline{v}(x) &= 0 \\
\overline{w}(x) &= 0 \\
\frac{dc}{dy} &= 0 \\
D_t &\gg D_{tx}
\end{align*}
\]

\[
\frac{dc}{dt} + \overline{u} \frac{dc}{dx} = D_{tz} \frac{\partial^2 c}{\partial z^2}
\]

Substitute Modified Reynold's Decomposition:
\[
\frac{d(\overline{c} + c')}{dt} + (\overline{u} + u') \frac{d(\overline{c} + c')}{dx} = D_{tz} \frac{\partial^2 (\overline{c} + c')}{\partial z^2} + D_{tx} \frac{\partial^2 (\overline{c} + c')}{\partial x^2}
\]

(1) Move coordinate system with mean flow:
\[
\begin{align*}
\xi &= x - \overline{u} t \\
t &= t \\
z &= z
\end{align*}
\]

Chain rule:
\[
\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} = -\overline{u} \frac{\partial}{\partial \xi} + \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z}
\]

Substitute:
\[
\frac{d(\overline{c} + c')}{dt} - \overline{u} \frac{d(\overline{c} + c')}{d\xi} + (\overline{u} + u') \frac{d(\overline{c} + c')}{d\xi} = D_z \frac{\partial^2 (\overline{c} + c')}{\partial z^2} + D_x \frac{\partial^2 (\overline{c} + c')}{\partial \xi^2}
\]
\[
\frac{\partial (\bar{c} + \bar{c}')}{\partial t} + u' \frac{\partial (\bar{c} + \bar{c}')}{\partial \xi} = D_z \frac{\partial^2 (\bar{c} + \bar{c}')}{\partial z^2} + D_x \frac{\partial^2 (\bar{c} + \bar{c}')}{\partial \xi^2}
\]

2. Take depth average:

\[
\bar{f} = \frac{1}{h} \int_0^h f(z) \, dz
\]

\[
\frac{\partial (\bar{c} + \bar{c}')}{\partial t} + \frac{\partial (u'\bar{c} + u'\bar{c}')}{\partial \xi} = D_z \frac{\partial^2 (\bar{c} + \bar{c}')}{\partial z^2} + D_x \frac{\partial^2 (\bar{c} + \bar{c}')}{\partial \xi^2}
\]

We can move \( u' \) inside since \( \nabla u' = 0 \) by continuity.

\[
\frac{\partial \bar{c}}{\partial t} + \frac{\partial (u'\bar{c})}{\partial \xi} = D_z \frac{\partial^2 \bar{c}}{\partial z^2} + D_x \frac{\partial^2 \bar{c}}{\partial \xi^2}
\]

\[\bar{c} \neq f(z).\]

\[
\frac{\partial \bar{c}}{\partial t} + \frac{\partial (\bar{w}c)}{\partial \xi} = 0 + D_x \frac{\partial^2 \bar{c}}{\partial \xi^2}
\]

3. We need a model for \( \bar{w}c' \). We do something quite clever: subtract depth-average equation from non-depth-average version.

\[
\frac{\partial (\bar{c} + \bar{c}')}{\partial t} + u' \frac{\partial (\bar{c} + \bar{c}')}{\partial \xi} = D_z \frac{\partial^2 \bar{c}'}{\partial z^2} \quad (\bar{c} \neq f(z))
\]

\[
- \frac{\partial \bar{c}}{\partial t} + \frac{\partial (\bar{w}c')}{\partial \xi} = 0 + D_x \frac{\partial^2 \bar{c}}{\partial \xi^2}
\]

\[
\frac{\partial \bar{c}'}{\partial t} + u' \frac{\partial \bar{c}}{\partial \xi} + u' \frac{\partial \bar{c}'}{\partial \xi} - \frac{\partial (\bar{w}c')}{\partial \xi} = D_z \frac{\partial^2 \bar{c}'}{\partial z^2} - D_x \frac{\partial^2 \bar{c}}{\partial \xi^2}
\]