Issue of Scale and Complexity

Real world problems are 3D and turbulent.

Our computer resources and complexity of input/output - often limit our analysis to 1D or 2D approximations with simple turbulent closures.

The trade-offs are done by performing scale analysis to determine the important scales in the problem.

The 3D transport equation with 1st order reaction:

\[
\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} + \frac{\partial (vC)}{\partial y} + \frac{\partial (wC)}{\partial z} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2} \pm k C
\]

3 dimensions: x, y, z
3 unit scales: mass (concentration), length, and time
3 processes: advection, diffusion, and reaction.

Select characteristic scales for normalization:

\[x = L_x X, \quad y = L_y Y, \quad z = L_z Z, \quad u = U U', \quad v = V V', \quad w = W W', \quad C = C_0 C'\]
For example,

$L_x, L_y, L_z$ can be the length to the point of interest in a river, width and depth of the river, respectively.

$U, V, W$ the $x$, $y$, $z$ components mean velocity in the river.

$C_0$ initial concentration

$T_x$ half-life

$T_0$ $L/U$, e.g., $L_x/U$ or diurnal cycle.

The choice depends on the problem.

Substituting into the transport equation (and divided by $C_0$):

$$\frac{1}{T_0} \frac{dc'}{dt'} + \frac{U}{L_x} \frac{\partial \left(u_c'\right)}{\partial x'} + \frac{V}{L_y} \frac{\partial \left(v_c'\right)}{\partial y'} + \frac{W}{L_z} \frac{\partial \left(w_c'\right)}{\partial z'}$$

$$= \frac{D_x}{L_x^2} \frac{\partial^2 c'}{\partial x'^2} + \frac{D_y}{L_y^2} \frac{\partial^2 c'}{\partial y'^2} + \frac{D_z}{L_z^2} \frac{\partial^2 c'}{\partial z'^2} + \frac{1}{T_x} k c'$$

Typically, we multiply the equation by $T_0$ to make it dimensionless.

Note

1. The characteristic scales may have different order of magnitude.

   e.g., $L_x \gg L_y$ or $T_x \gg T_0$

2. All the dimensionless variables have order of magnitude $O(1)$.

   e.g., $x' = \frac{x}{L_x}$; $y' = \frac{y}{L_y}$

   $k' = \frac{k}{T_x}$; $t' = \frac{t}{T_0}$

   \[\uparrow\] because normalized by its own scale!
Since all the normalized variables are of order $O(1)$, the relative magnitude of each term depends on the added scalers.

\[ \frac{\text{longitudinal advection}}{\text{lateral advection}} = \frac{U/L_y}{V/L_y} = \frac{U}{V} \frac{L_y}{L_x} \]

Typically, \( \text{longitudinal advection} \gg \text{lateral advection in a river.} \) This implies \( \frac{U}{V} \frac{L_y}{L_x} \gg 1 \)

Another example:

\[ \frac{\text{longitudinal diffusion}}{\text{longitudinal advection}} = \frac{D^2/L_y}{U/L_x} = \frac{D_x}{L_x U} \quad (= Pe) \]

This is the Peclet number. Large \( Pe \rightarrow \) diffusion dominant.
Small \( Pe \rightarrow \) advection dominant.

Consider a 1D steady-state model with die-off:

\[ u \frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} - k C \]

This equation implies:

\[ \frac{\partial C}{\partial t} \ll u \frac{\partial C}{\partial x} \quad \text{or} \quad \frac{1}{U/L_x} = \frac{L_x}{U T_0} \ll 1 \]
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{v}{\frac{\partial c}{\partial y}} \ll \frac{u}{\frac{\partial c}{\partial x}} \quad \text{or} \quad \frac{V/L_y}{U/L_x} = \frac{V/L_y}{U/L_x} \ll 1 \\
\frac{w}{\frac{\partial c}{\partial y}} \ll \frac{u}{\frac{\partial c}{\partial x}} \quad \text{or} \quad \frac{W/L_3}{U/L_x} = \frac{W/L_x}{U/L_3} \ll 1 \\
\frac{\frac{\partial^2 c}{\partial y^2}}{D_y} \ll \frac{\frac{\partial^2 c}{\partial x^2}}{D_x} \quad \text{or} \quad \frac{D_y/L_y}{D_x/L_x} = \frac{D_y/L_y}{D_x/L_x} \ll 1 \\
\frac{\frac{\partial^2 c}{\partial y^2}}{D_y} \ll \frac{\frac{\partial^2 c}{\partial x^2}}{D_x} \quad \text{or} \quad \frac{D_y/L_y}{D_x/L_x} = \frac{D_y/L_y}{D_x/L_x} \ll 1 \\
\end{array} \right.
\end{align*}
\]

\[P_e = \frac{D_x/L_x^2}{U/L_x} \approx 1\]

\[\frac{kC}{u \frac{\partial c}{\partial x}} = \frac{1}{U/L_x} = \frac{L_x}{U/L_x} \approx 1\]