Diffusion Equation:

Fick's Law tells us the rate of transport of a substance due to diffusion, but it doesn't tell us how $C$ changes in time and space due to this diffusive flux.

For that, we need to invoke conservation of mass. Consider a small control volume:

\[
\frac{dM}{dt} = m_{\text{in}} - m_{\text{out}} = \delta m
\]

Consider only the $x$-direction:

\[
\delta m_x = \delta y \delta z \left. q_x \right|_0 - \left. \delta y \delta z q_x \right|_2
\]

Apply Fick's Law:

\[
\delta m_x = \delta y \delta z \left( -D \frac{\partial C}{\partial x} \right)_1 - \left( -D \frac{\partial C}{\partial x} \right)_2
\]
Use Taylor Series Expansion:

\[ f(x_2) = f(x_1) + \frac{df}{dx}\bigg|_{x_1} \delta x + \ldots + \]

\[ \text{higher order terms of order } O(\delta x^2) \]

For \( \frac{dc}{dx} \):

\[ \frac{dc}{dx}\bigg|_{x_2} = \frac{dc}{dx}\bigg|_{x_1} + \frac{d}{dx}\left( \frac{dc}{dx}\bigg|_{x_1} \right) \delta x + O(\delta x^2) \]

Substitute into mass conservation:

\[ S_{1x} = -D \delta y \delta z \left[ \frac{dc}{dx}\bigg|_{x_1} - \frac{dc}{dx}\bigg|_{x_1} - \frac{d^2c}{dx^2}\bigg|_{x_1} \delta x - O(\delta x^2) \right] \]

Neglect.

\[ S_{1x} = D \delta x \delta y \delta z \frac{d^2c}{dx^2} \]

Also note:

\[ M = C \delta x \delta y \delta z \]

Combine All 3 Dimensions:

\[ \delta x \delta y \delta z \frac{dc}{dt} = D \delta y \delta z \delta x \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) \]

\[ \frac{dc}{dt} = D \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) \]

\[ \frac{dc}{dt} = D \nabla^2 c \] or \[ \frac{dc}{dt} = D \frac{dc}{dx^2} \]
Diffusion Equation:

\[
\frac{dc}{dt} = D \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right)
\]

Time rate of change of \( c \).

1 initial condition
2 boundary conditions per dimension.

"Parabolic" Partial differential Equation.
Same as the heat equation.

Example: 1D diffusion in a pipe:

\[\begin{array}{c}
\text{C}_1 \\
\text{fixed conc.}
\end{array}\quad L \quad \begin{array}{c}
\text{C}_2 \\
\text{fixed conc.}
\end{array}\quad x
\]

Steady state.

1D: \( \frac{dc}{dy}, \frac{dc}{dz} = 0 \)

Steady state: \( \frac{dc}{dt} = 0 \)

Boundary Conditions:

\( C(0) = C_1 \)
\( C(L) = C_2 \)
Solution:

\[ 0 = D \frac{\partial^2 c}{\partial x^2} \]

\[ \frac{\partial^2 c}{\partial x^2} = 0 \]

\[ \int \frac{\partial^2 c}{\partial x^2} \, dx = \frac{\partial c}{\partial x} + c_1' = 0 \]

\[ \int \left( \frac{\partial c}{\partial x} + c_1' \right) \, dx = c + c_1'x + c_2 = 0 \]

\[ c = c_1'x + c_2' \]

Apply B.C.s:

\[ c(0) = c_1 = c_1'(0) + c_2' \quad \Rightarrow \quad c_2' = c_1 \]

\[ c(L) = c_2 = c_1'(L) + c_1 \quad \Rightarrow \quad c_1' = \frac{c_2 - c_1}{L} \]

\[ c(x) = \frac{c_2 - c_1}{L} x + c_1 \]