**1D Example:** Diffusion in an infinite pipe.

M: well-mixed in y and z.
A: pipe cross-section.

**Notes:**
1D problem: concentration units \( \frac{M}{L} \)
(per unit area).
Initial concentration infinite.
No flow in pipe.

** Governing Equation:**

\[
\frac{\partial c}{\partial t} = D_m \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right)
\]

\(O:\) well-mixed in y
\(O:\) well-mixed in z.

\[
\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}
\]

So, we need one initial condition and two boundary conditions.

**Boundary Conditions:**

No mass can wonder all the way to infinity, so the two boundary conditions are

\[C(\infty, t) = 0\]
\[C(-\infty, t) = 0\]
Initial Condition:

To specify an instantaneous, point source we use a special function called a "Dirac Delta Function":

\[ \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{otherwise} \end{cases} \]

And it also has the property:

\[ \int_{-\infty}^{\infty} \delta(x) \, dx = 1. \]

So, it has finite area, but an infinite value.

Our initial condition becomes:

\[ c(x, 0) = \frac{M}{A} \delta(x) \]

We can check if this works by finding the total mass injected:

\[
M = \int_{V} C(x^*, t) \, dV \quad \text{cylindrical coordinates.}
\]

\[
= \int_{-\infty}^{\infty} \int_{0}^{a} C(x^*, t) \ 2\pi r \, dr \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{0}^{a} \frac{M}{A} \delta(x) \ 2\pi r \, dr \, dx
\]

\[
= \int_{-\infty}^{\infty} \frac{M}{A} \delta(x) \int_{0}^{a} 2\pi r \, dr \, dx
\]

\[
M = \int_{-\infty}^{\infty} M \delta(x) \, dx = M \int_{-\infty}^{\infty} \delta(x) \, dx = M \checkmark \]
So, we can summarize our problem as:

\[ \frac{dc}{dt} = D \frac{d^2c}{dx^2} \]

B.C.: \[ c(\infty, t) = 0 \]
\[ c(-\infty, t) = 0 \]

I.C.: \[ c(x, 0) = \frac{M}{A} \delta(x) \]

Auxillary Condition: \[ M = \int_{-\infty}^{\infty} c(x, t) \, dx \]

**Similarity Solution:**

To solve a PDE you generally must first convert it to an ODE.

Many methods work. Fourier Transform is in Appendix A.

We’ll use a method that relies on dimensional analysis.

Dependent variable we want to predict:
\[ c(x, t) \]

Independent variables \( c \) depends on:
\[ \frac{M}{A} \quad [\text{M/L}] \]
\[ D \quad [\text{L}^2/\text{T}] \]
\[ x \quad [\text{L}] \]
\[ t \quad [\text{T}] \]

So, we have 5 parameters: \( c, \frac{M}{A}, D, x, t \).
And 3 dimensions: L, T, M.

\[ \Rightarrow 5 - 3 = 2 \text{ dimensionless groups.} \]
When forming dimensionless groups, try to only use $C$ once:

dependent variable $\Pi_1 = \frac{C}{M/\sqrt{NA DT}}$

$\Pi_2 = \frac{X}{\sqrt{D T}}$

Buckingham $\Pi$ theorem gives:

$\Pi_1 = f(\Pi_2)$

$\frac{AC\sqrt{D T}}{M} = f\left(\frac{X}{\sqrt{D T}}\right)$

Rearrange:

$C(x,t) = \frac{M}{AN\sqrt{D T}} f\left(\frac{X}{\sqrt{D T}}\right)$

So, this will be the solution to our problem. To use the solution, we must find out what $f$ is.

1. Do experiment and fit $f$ to data.
2. Solve for $f$ using governing equation.

We will substitute this $C(x,t)$ into our problem and solve for $f$.

Define new variable: $\eta = \frac{X}{\sqrt{D T}}$. (called similarity variable).

Solve for $f(\eta)$