From dimensional analysis:

\[ \pi_1 = \frac{C(A \sqrt{Dt})}{M} \]

\[ \pi_2 = \frac{x}{\sqrt{Dt}} \]

Buckingham \( \Pi \) theorem:

\[ C = \frac{M}{A \sqrt{Dt}} \cdot f \left( \frac{x}{\sqrt{Dt}} \right) \]

Rename similarity variable:

\[ \eta = \frac{x}{\sqrt{Dt}} \]

Note:

\[ \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{Dt}} \quad \frac{\partial \eta}{\partial t} = -\frac{x}{2\sqrt{Dt}} \cdot \frac{1}{t} = -\frac{\eta}{2t} \]

Substitute similarity solution into governing equation:

\[ \frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \]

This is unknown

\[ \frac{\partial C}{\partial t} = f(\eta) \frac{\partial}{\partial t} \left( \frac{M}{AN^2Dt} \right) + \frac{M}{AN^2Dt} \frac{\partial f}{\partial \eta} \left( \frac{\partial \eta}{\partial t} \right) \]

This is known.

\[ = \frac{M}{A} (-\frac{1}{2}) \frac{1}{\sqrt{Dt}^3} \cdot f + \frac{M}{AN^2Dt} \frac{\partial f}{\partial \eta} \left( -\frac{\eta}{2t} \right) \]

\[ = \frac{M}{AN^2Dt} \left( -\frac{1}{2t} \right) \left[ f + \eta \frac{\partial f}{\partial \eta} \right] . \]
\[ \frac{dc}{dx} = \frac{M}{A \sqrt{Dt}} \frac{df}{d\eta} \frac{d\eta}{dx} \]

\[ = \frac{M}{A \sqrt{Dt} \sqrt{Dt}} \frac{df}{d\eta} \]

\[ \frac{d^2c}{dx^2} = \frac{d}{dx} \left( \frac{dc}{dx} \right) = \frac{d}{dx} \left( \frac{M}{A \sqrt{Dt}} \frac{1}{\sqrt{Dt}} \frac{df}{d\eta} \right) \]

\[ = \frac{M}{ADt \sqrt{D} \sqrt{D}} \frac{d^2f}{d\eta^2} \frac{1}{\sqrt{Dt}} \]

\[ = \frac{M}{ADt \sqrt{D} \sqrt{D}} \cdot \frac{d^2f}{d\eta^2} \]

Combining:

\[ \frac{dc}{dt} = D \frac{d^2c}{dx^2} \]

\[ \frac{M}{A \sqrt{Dt} \sqrt{D}} \left( -\frac{1}{2x} \right) \left( f + \eta \frac{df}{d\eta} \right) = \frac{D}{ADt \sqrt{D}} \frac{d^2f}{d\eta^2} \]

\[ \frac{d^2f}{d\eta^2} + \frac{1}{2} \left( f + \eta \frac{df}{d\eta} \right) = 0 \]

*Function of one variable → this has become an ODE.*

\[ \frac{d^2f}{d\eta^2} + \frac{1}{2} \left( f + \eta \frac{df}{d\eta} \right) = 0 \]

*non-linear.*
Note:
\[
\frac{d(f\eta)}{d\eta} = f \frac{df}{d\eta} + \eta \frac{df}{d\eta}
\]
\[
= f + \eta \frac{df}{d\eta}
\]
Substituting:
\[
\frac{d^2f}{d\eta^2} + \frac{1}{2} (f + \eta \frac{df}{d\eta}) = 0
\]
\[
\frac{d^2f}{d\eta^2} + \frac{1}{2} \frac{d(f\eta)}{d\eta} = 0
\]
\[
\frac{d}{d\eta} \left( \frac{df}{d\eta} + \frac{1}{2} f\eta \right) = 0
\]

Boundary, initial and auxiliary condition:

Auxiliary condition:
\[
\frac{M}{A} = \int_{-\infty}^{\infty} c(x,t) \, dx
\]
\[
= \int_{-\infty}^{\infty} \frac{M}{ANt} \cdot f(\eta) \, dx
\]

Change of variable: \( \eta = \frac{x}{\sqrt{At}} \)
\[
d\eta = \frac{dx}{\sqrt{At}}
\]
\[
\frac{M}{A} = \int_{-\infty}^{\infty} \frac{M}{ANt} \cdot f(\eta) \sqrt{At} \, d\eta
\]
\[
\int_{-\infty}^{\infty} f(\eta) \, d\eta = 1.
\]
Boundary Conditions:

\[ C(\pm \infty, t) = 0 \]

\[ \frac{M}{A \sqrt{D \ell}} f \left( \frac{x}{\sqrt{D \ell}} \right) = 0 \]

\[ \frac{M}{A \sqrt{D \ell}} f \left( \frac{\pm \infty}{\sqrt{D \ell}} \right) = 0 \]

\[ f(\pm \infty) = 0 \]

Initial Condition:

\[ C(x, 0) = \frac{M}{A} \delta(x) \]

\[ \frac{M}{A \sqrt{D \ell}} f \left( \frac{x}{\sqrt{D \ell}} \right) = \frac{M}{A} \delta(x) \]

\[ f \left( \frac{x}{\sqrt{D \ell}} \right) = \sqrt{D \ell} \delta(x) \]

\[ \text{if } x > 0, \quad +\infty \quad 0 \quad 0 \quad 0 \]
\[ \text{if } x < 0, \quad -\infty \quad 0 \quad 0 \quad 0 \quad 0 \]

\[ f(\pm \infty) = 0 \]

Because the initial and boundary conditions are space and time conditions and \( \eta \) unifies space and time, the boundary and initial conditions are identical in \( \pm \infty \).

Integrate governing equation once:

\[ \frac{df}{d\eta} + \frac{1}{2} f \eta = c_0 \]

\( c_0 = 0 \) satisfies \( f(\pm \infty) = 0 \) ... we'll check this after we get \( f \).
Continuing:
\[
\frac{df}{d\eta} = -\frac{1}{2} f \eta
\]
\[
\frac{df}{f} = -\frac{1}{2} \eta \, d\eta
\]
Integrate both sides:
\[
\ln(f) = -\frac{\eta^2}{4} + C_1^*
\]
Take exponential of both sides:
\[
f = \exp\left(-\frac{\eta^2}{4} + C_1^*\right)
\]
\[
= C_1 \exp\left(-\frac{\eta^2}{4}\right)
\]
Find value of $C_1$ from auxiliary condition:
\[
1 = \int_{-\infty}^{\infty} f(\eta) \, d\eta
\]
\[
= \int_{-\infty}^{\infty} C_1 \exp\left(-\frac{\eta^2}{4}\right) \, d\eta
\]
Define new variable:
\[
\xi = \frac{\eta^2}{4}
\]
\[
2 \xi \, d\xi = \frac{1}{4} \eta \, d\eta
\]
\[
\frac{2^2}{2} \, d\xi = \frac{2^2}{4} \, \eta \, d\eta \rightarrow 2 \, d\xi = d\eta
\]
Substituting:

\[ 1 = \int \limits_{-\infty}^{\infty} c_1 \exp\left(-\xi^2\right) 2 \, d\xi \]

\[ = 2c_1 \int \limits_{-\infty}^{\infty} \exp\left(-\xi^2\right) \, d\xi \]

Table:

\[ \int \limits_{0}^{\infty} \exp\left(-ax^2\right) \, dx = \frac{1}{2a} \sqrt{\pi} \]

\[ a = 1 \]

\[ \int \limits_{0}^{\infty} \exp\left(-x^2\right) \, dx = \frac{\sqrt{\pi}}{2} \]

\[ \int \limits_{-\infty}^{0} \exp\left(-x^2\right) \, dx = \int \limits_{0}^{\infty} \exp\left(-x^2\right) \, dx \]

\[ \therefore \int \limits_{-\infty}^{\infty} \exp\left(-\xi^2\right) \, d\xi = \sqrt{\pi} \]

Solve for \( c_1 \):

\[ 1 = 2c_1 \sqrt{\pi} \]

\[ c_1 = \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{4\pi}} \]

Combining:

\[ f = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4Dt}\right) \]
Substitute \( f \to C \):

\[
C(x,t) = \frac{M}{A \sqrt{4\pi DT}} f \left( \frac{x}{\sqrt{4DT}} \right)
\]

\[
C(x,t) = \frac{M}{A \sqrt{4\pi DT}} \exp \left( -\frac{x^2}{4DT} \right)
\]

Recall Gaussian distribution:

\[
P(x,\sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{x^2}{2\sigma^2} \right)
\]

Compare \( \frac{CA}{M} \) to this... what is \( \sigma \)?