Review: Diffusion \( \frac{\partial C}{\partial t} = D \nabla^2 C \)

(point-source solution (for 1D prob.))

\[
C(x, t) = \frac{M}{A \sqrt{4 \pi D t}} \exp \left[ -\frac{(x-x_0)^2}{4Dt} \right]
\]

\( x_0 \): injection point

What if an uniform velocity is added? (therefore w/ advection)

It could look like

![Diffusion only and Advection + Diffusion](image)

Center of mass at \( x_c = x_0 + ut \)

If we move the coordinate system w/ \( x_c \):

\[
\dot{x} \equiv x - x_c = x - x_0 - ut
\]

The case may become the point-source problem. We can guess the solution would be

\[
C(x, t) = \frac{M}{A \sqrt{4 \pi D t}} \exp \left[ -\frac{(x-x_0-ut)^2}{4Dt} \right]
\]
*Show A-D animation*

Is this solution correct? What if $u$ is constant?
We need a general equation for advective diffusion

Consider a small control volume with crossflow

\[
\frac{\partial M}{\partial t} = \sum m_{in} - \sum m_{out}
\]

Let's consider only the $x$-direction.

**Diffusive mass flux (from Fick's Law):**

\[
m_d = A \frac{\partial C}{\partial x} = -AD \frac{\partial C}{\partial x}
\]

**Advection mass flux:**

\[
m_a = ACu
\]

Thence the A-D flux is

\[
J_x = \frac{m_a + m_d}{A} = C \nu - D \frac{\partial C}{\partial x}
\]

or \[\vec{J} = C \vec{u} - D \nabla C \] for 3D.
\[ \delta m_x = \left( J_{x1} - J_{x2} \right) \delta y \delta z \]
\[ = [ (CU - D \frac{\partial C}{\partial x})_1 - (CU - D \frac{\partial C}{\partial x})_2 ] \delta y \delta z \]

Recall Taylor Series:
\[ f(x + \Delta x) = f(x) + \frac{\partial f}{\partial x} \Delta x + H.O.T. \]

or
\[ f|_2 - f|_1 = \frac{\partial f}{\partial x}|_1 \Delta x + H.O.T. \]

\[ CU|_1 - CU|_2 = -\frac{\partial (CU)}{\partial x}|_1 \Delta x \]
\[ D \frac{\partial C}{\partial x}|_2 - D \frac{\partial C}{\partial x}|_1 = \frac{\partial}{\partial x} \left( D \frac{\partial C}{\partial x} \right)|_1 \Delta x = D \frac{\partial^2 C}{\partial x^2} \Delta x \]

(for constant D)

So
\[ \delta \dot{m}_x = \left[ -\frac{\partial (CU)}{\partial x} + D \frac{\partial^2 C}{\partial x^2} \right] \delta x \delta y \delta z \]

Similarly
\[ \delta \dot{m}_y = \left[ -\frac{\partial (CV)}{\partial y} + D \frac{\partial^2 C}{\partial y^2} \right] \delta x \delta y \delta z \]
\[ \delta \dot{m}_z = \left[ -\frac{\partial (CW)}{\partial z} + D \frac{\partial^2 C}{\partial z^2} \right] \delta x \delta y \delta z \]

Recall
\[ M = C \delta x \delta y \delta z \]

Substitute into conservation of mass
\[ \frac{\partial C}{\partial t} \delta x \delta y \delta z = \delta \dot{m}_x + \delta \dot{m}_y + \delta \dot{m}_z \]
Finally we obtain

\[ \frac{\partial C}{\partial t} + \nabla \cdot (\vec{u} C) = \nabla^2 C \quad \text{A-D equation} \]

(or \[ \frac{\partial C}{\partial t} + \frac{\partial (\vec{u} C)}{\partial x_i} = D \frac{\partial^2 C}{\partial x_i^2} \quad \text{in tensor form} \)

* For incompressible fluid (such as water, but not air)

\[ \nabla \cdot (\vec{u} C) = \vec{u} \nabla C + C \vec{u} \cdot \vec{u} \]

\[ = 0 \quad \text{for incompressible fluid} \]

\[ = \vec{u} \nabla C \]

\[ \frac{\partial C}{\partial t} + \vec{u} \nabla C = \nabla^2 C \quad \text{A-D equation for incompressible fluid} \]

(or \[ \frac{\partial C}{\partial t} + \vec{u}_i \frac{\partial C}{\partial x_i} = D \frac{\partial^2 C}{\partial x_i^2} \quad \text{in tensor form} \)
Derivation of $\nabla \cdot \mathbf{u} = 0$

Recall conservation of mass in integral form (equation 5.5 in Munson et al. 2002)

$$\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho \mathbf{u} \cdot \mathbf{n} \, dA = 0$$

Let's fix the control volume

$$\frac{\partial}{\partial t} \int_{CV} \rho \, dV = \int_{CV} \frac{\partial \rho}{\partial t} \, dV \quad \text{(1)}$$

Gauss's theorem:

$$\int_{A} \mathbf{a} \cdot \mathbf{n} \, dA = \int_{V} \nabla \cdot \mathbf{a} \, dV$$

$$\int_{CS} \rho \mathbf{u} \cdot \mathbf{n} \, dA = \int_{CV} \nabla \cdot (\rho \mathbf{u}) \, dV \quad \text{(2)}$$

$$(1) + (2): \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{continuity equation}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0$$

if $\rho = \text{constant}$ (e.g., an incompressible fluid)

$$\frac{\partial \rho}{\partial t} = 0 \quad \nabla \rho = 0$$

$\Rightarrow \quad \nabla \cdot \mathbf{u} = 0$ \quad \text{continuity equation for incompressible flow (w/ a constant $\rho$)}