1 Stream function

The stream function for a certain 2D, incompressible flow field is

\[ \psi = 10y + e^{-y} \sin x \]  

(1)

Is this an irrotational flow field? Justify your answer with the necessary calculations.

2 Velocities in Flows around Objects

Show that the maximum tangential velocity for the streaming flow past a cylinder is twice the free-stream value and determine the location of the maximum velocity. Repeat your calculations when the cylinder is rotating. Relate the angular velocity of the cylinder wall to the increase in maximum velocity over the non-rotating cylinder case.

3 Tornados

A tornado is sometimes represented by the superposition of a counter-clockwise rotating vortex and a sink.

1. Write down the expressions for the complex potential \( F \) and complex velocity \( W \) for the vortex strength \( \Gamma \) and sink magnitude \( m \).

2. Using polar coordinates, derive an expression for the pressure in a tornado versus distance \( r \) from the center.
3. From the relationship
\[ \oint W dz = \Gamma + iQ \] \hspace{1cm} (2)
where \( \Gamma \) is the circulation and \( Q \) is the flow rate through the contour, show that the tornado has circulation equal to the vortex strength and flow rate equal to the negative of the sink strength.

4 Stream lines

Take a 2D plane source of strength \( m_1 \) at point \((-a, 0)\), a 2D plane sink of strength \( m_2 \) at point \((a, 0)\), and superpose a uniform stream \( U \) directed along the \( x \)-axis.

1. Write the complex potential and complex velocity for this flow field. From these complex functions, extract the equations for the velocity components \( u \) and \( v \), the velocity potential \( \phi \), and the stream function \( \psi \).

2. Use a computer program to plot several streamlines for the case \( m_1 = 2, m_2 = 1, a = 1, \) and \( U = 0.5 \). Plot lines for several values of \( \psi \) so that the flow field is well-resolved. Determine which streamlines are also the surface of a solid body.

3. Use your plotting algorithm to plot the flow field when \( m_1 = m_2 = 1 \). Again determine which streamlines are the surface of a solid body.

4. Use your solutions in part 1 to compute the locations of the stagnation points on the \( x \)-axis when \( m_1 = m_2 \).

5. For the case \( m_1 = m_2 \), show that the maximum width \( h \) in the \( y \)-direction is given by
\[ h = a \cot \left( \frac{\pi U h}{m} \right) \] \hspace{1cm} (3)

6. How is the stagnation pressure defined and how could it be calculated for this flow?

7. Use a computer program to calculate the pressure distribution on the surface of the object for \( m_1 = m_2 = 1 \) using Bernoulli’s equation and setting the pressure far from the source as \( p(\pm \infty) = 0 \). Where are the maximum and minimum pressures on the object?

8. Use Blasius integral laws to show that the net force on this object when \( m_1 = m_2 = 1 \) is \( \vec{F} = (0, 0) \).
5 Conformal Mapping

An important transformation for 2D potential flow is the Joukow ski transformation, given by

\[ z = \zeta + \frac{c^2}{\zeta} \]  

Consider a circle in the complex \( \zeta \)-plane as shown in Figure 1.

Figure 1: \( \zeta \)-plane object for Joukowski transformation.

1. Write the equation for the circle using the parameters shown in the figure.

2. Use the Joukowski transformation to obtain an object in the \( z \) plane. Do this using a computer program to plot a profile of the object in the \( z \) plane for \( c = 1, \delta = 100^\circ \).

3. Write the complex potential in the \( \zeta \)-plane for the circular cylinder shown in the figure with a uniform flow of \( u_\infty \) in the \( \xi \)-direction.

4. Transform the solution in part 3 from the \( \zeta \)-plane to the \( z \)-plane to find \( F(z) \). (Hint: expand Equation (4) to obtain a quadratic equation for \( \zeta \) as a function of \( z \). Take the positive root in the quadratic equation. Substitute this expression into the solution from part 3.)

5. Manipulate \( F(z) \) to find the complex velocity \( W(z) \) and form the integrand for Blasius integral law \( W^2 \). Determine the locations of the poles of \( W^2 \). Based on your analysis and Blasius integral laws, is it possible that there is a net force on the object in the \( z \)-plane?