1 Falling Flat Plate

We return to the problem from the exam of trying to lift a flat plate off of a solid surface. In the exam, the velocity within the gap was provided. In this assignment, we derive expressions for the velocity following two approximations. For the following problems, refer to Figure 1

![Figure 1: Sketch of the case of lifting a flat plate of finite lateral extent off of a solid surface.](image)

1.1 Ideal flow results

The flow created by a flat plate being lifted from a wall with constant velocity $V_0$ is modeled first as a 2D ideal flow (ignore gravity). The width of the plate is $2L$, the height between the plate and the wall is $b(t)$. For $b \ll L$ the horizontal velocity in the gap can be assumed to be uniform in $y$: $u = u(x)$. By symmetry, on the centerline $x = 0$, $u(x = 0) = 0$. Note that because of the velocity $V_0$ of the rising plate, the vertical velocity inside is not zero: $v \neq 0$. 
1. Apply conservation of mass in a control volume $ABCD$ from $x = 0$ to $x = x$ to find the horizontal velocity component as a function of $V_0$, $b$, and $x$.

2. Using the result above, apply the differential form of the continuity equation and the proper boundary condition(s) to show that the vertical velocity varies linearly with $y$ and is independent of $x$. Write down $v(y)$ in terms of $V_0$, $b$, and $y$.

3. Apply the unsteady form of the Bernoulli equation to obtain an expression for the pressure in the gap as a function of $V_0$, $b$, $x$, and $\rho$. To evaluate the Bernoulli constant, set the pressure outside the gap to zero at an arbitrary point where the outer velocity is also zero.

4. In the exam, we integrated the pressure distribution to find the force exerted on the plate by the fluid to be

$$F_y = -\frac{2\rho V_0^2 L^3}{3b^2}$$

(1)

For a plate falling under its own weight, we can replace $F_y$ by the buoyant weight of the plate $W$ and rearrange this equation to solve for the fall velocity $V_0$. Note that the plate exerts an equal and opposite force on the water and that the velocity is negative when the plate is falling. Recalling that $\frac{db}{dt} = V_0$, use these equations to find an expression for $b(t)$ for the free-falling plate. Assume that the plate starts from rest at a height $b_0$ at the time $t = 0$ and that the plate falls slowly enough that the steady flow solution is valid at each instant of its descent.

1.2 Viscous flow results

Going back to the case of lifting the plate, when we first try to lift the plate, the flow is very slow and the boundary layers are of similar size to the gap, so that a viscous flow solution is preferred to the one derived above.

1. Based on the following assumptions, derive the exact governing equations for the flow in the gap:
   - Fluid is incompressible and Newtonian
   - Extent of the object into the page is large and there is no driving force in the $z$ direction such that $w = 0$ and $\partial/\partial z = 0$

2. Next, non-dimensionalize the governing equations by introducing the following scales

$$u = U u^* \quad v = V v^* \quad x = L x^* \quad y = b y^* \quad t = (b/V_0) t^* \quad p = P p^*$$

(2)
where the scales are defined in Figure 1. Use the conservation of mass to establish a
relationship between $U$ and $V$, and take the vertical scale $V = V_0$, as the velocity of the
plate. Non-dimensionalize the momentum equations so that the coefficient multiplying
the pressure terms have the same value.

3. Introduce the following assumptions and delete the negligible terms from the non-
dimensional governing equations
   • Narrow gap: $b \ll L$.
   • Rate of separation $V_0$ is slow: $Re = V_0 b / \nu \ll 1$.

Once the important terms in the equations are obtained, write the resulting set of
approximate equations back in dimensional form. Solve the remaining parts in dimen-
sional form.

4. From the approximate equations, show that the pressure gradient can only be a function
   of $x$; hence, show that $dp/dx = f(x)$.

5. Integrate the approximate momentum equation in the $x$-direction and apply the ap-
   propriate boundary conditions to obtain an expression for the horizontal velocity com-
   ponent in terms of $\mu$, $dp/dx$, $y$, and $b$.

6. Return to the control volume $ABCD$ used in the inviscid solution. Use the conservation
   of mass over this control volume to obtain an equation for $dp/dx$. Note that $u$ is now a
   function of $y$ that must be integrated over the appropriate face(s) of the control volume
to determine the net inflow. Solve the resulting equation to obtain an expression for
the pressure as a function of $\mu$, $V_0$, $b$, $L$, and $x$. Use the boundary condition that the
pressure is equal to zero at $x = L$ to evaluate the integration constant when solving
for the pressure.

7. Substitute your expression for $dp/dx$ obtained in the last step into the equation for
the horizontal velocity. Then use the differential form of the conservation of mass to
obtain an expression for the vertical velocity as a function of $V_0$, $b$, $x$, and $y$. Why do
you think the vertical and horizontal velocities are independent of the viscosity?

8. Integrate the pressure over the surface of the plate at $y = b$ to obtain an expression
for the force exerted on the plate by the water. Assume that the pressure everywhere
outside the gap is zero.

9. Which solution gives the greater adhesion force: the inviscid or the viscous flow solu-
tion?

10. Follow the procedure used in the inviscid solution to find a viscous-flow expression for
the gap height as a function of time for the case of the free-falling plate. Which case
falls faster: the inviscid or the viscous flow solution?
2 Liquid Water Flow along an Icicle

Short et al. (Short, Baygents, & Goldstein, “A free-boundary theory for the shape of the ideal dripping icicle,” *Physics of Fluids* 18, 083101, 2006) analyze the flow of water down an icicle in order to obtain a non-dimensional relationship for the geometry of icicles. The following paragraph is taken from their paper with a slight change in the names of the variables so that it is more consistent with our notation:

We first consider the water layer flowing down the surface of a growing icicle to set some initial scales. The volumetric flow rate $Q$ over icicles is typically on the order of tens of milliliters per hour ($\approx 0.01 \text{ cm}^3/\text{s}$), and icicle radii are usually in the range of 1 – 10 cm. To understand the essential features of the flow, consider a cylindrical icicle of radius $R_0$, over the surface of which flows an aqueous film of thickness $h$ (refer to Figure 2(a)). Since $h \ll R_0$ over nearly the entire icicle surface, the velocity profile in the layer may be determined as that flowing on a flat surface. Furthermore, we expect the Reynolds number to be low enough that the Stokes approximation is valid [e.g., $Re \sim O(1)$ or less]. If $y$ is a coordinate normal to the surface and $\gamma$ is the angle that the icicle surface makes with respect to the horizontal (see Figure 2(b)), then the Stokes equation for gravity-driven flow is $\nu \frac{d^2 u}{dy^2} = g \sin \gamma$, where $g$ is the gravitational acceleration and $\nu = 0.01 \text{ cm}^2/\text{s}$ is the kinematic viscosity of water. Enforcing no-slip and stress-free boundary conditions at the solid-liquid and liquid-air interfaces, the thickness is

$$h = \left( \frac{3Q\nu}{2\pi g R_0 \sin \gamma} \right)^{1/3} .$$

Using typical flow rates and radii, we deduce a layer thickness that is tens of
microns and surface velocities \( u_s \approx (gh^2/2\nu) \sin \gamma \) below several mm/s, consistent with known values, yielding \( Re = 0.01 - 0.1 \), well in the laminar regime as anticipated. At distances from the icicle tip comparable to the capillary length (several millimeters), the complex physics of pendant drop detachment takes over and the thickness law Equation (3) ceases to hold.

Verify their analysis by working through the following steps.

1. Write the exact equations for steady, gravity-driven, sheet flow down a cylinder using cylindrical coordinates. Simplify the equations for steady flow and the given geometry. Gravity is the only driving force; for now, leave this body force as \( \vec{f} = (f_R, 0, f_z) \).

2. Use the assumption of shallow sheet flow \( (h \ll R_0) \) to show that the equations from step one can be written in the form for two-dimensional flow over a flat plate. That is, show that the exact equations may be approximated by

\[
\frac{\partial u_R}{\partial R} + \frac{\partial u_z}{\partial z} = 0 \quad (4)
\]

\[
u \frac{\partial u_R}{\partial R} + u_z \frac{\partial u_R}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial R} + \nu \left( \frac{\partial^2 u_R}{\partial R^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + f_R \quad (5)
\]

\[
u \frac{\partial u_z}{\partial R} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 u_z}{\partial R^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + f_z \quad (6)
\]

3. Use the coordinate system in Figure 2(b) to convert the equations in step two to a set of equations in the Cartesian coordinate system \( x - y \) for the flow over a flat plate. Substitute the correct form of the gravity terms based on the geometry in the figure.

4. Use scale analysis and the assumptions of shallow flow \( (h \ll L) \), Stokes approximation \( (Re_h \leq 1) \), and a shallow slope \( ((\pi/2 - \gamma) \ll 1) \) to obtain the following set of approximate governing equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (7)
\]

\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = g \cos \gamma \quad (8)
\]

\[
\nu \frac{\partial^2 u}{\partial y^2} = g \sin \gamma \quad (9)
\]

5. Solve these simplified equations for the \( u \)-component velocity to obtain

\[
u \left( \frac{y^2}{2} - hy \right) \quad (10)
\]
by applying a no slip boundary condition at the icicle wall \( u(0) = 0 \) and a shallow-flow stress-free boundary condition at the free surface \( du(h)/dy = 0 \).

6. To obtain an equation for the net flux \( Q \) over the icicle under the assumption \( h \ll R_0 \), we may use the solution for flow over a flat plate just obtained as follows

\[
Q(x) = 2\pi R_0(x) \int_0^h u\,dy
\]  

(11)

Solve this equation for the thickness \( h \) to obtain the expression in Equation (3).

7. From real icicles, we measure the net flux to be \( Q \approx 0.01 \text{ cm}^3/\text{s} \), a typical radius to be \( R_0 \approx 1 - 10 \text{ cm} \), and a typical wall angle as \( \gamma \approx 5^\circ \). Using these data, verify the assumptions in this derivation by showing that \( h \ll R_0 \) and \( Re_h \leq 1 \).