1 Energy equation

The momentum equation for an incompressible fluid in indicial notation can be given by

\[ \rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i \]  

(1)

where \( \sigma_{ij} = -p\delta_{ij} + \tau_{ij} \) represents the sum of the pressure tensor \(-p\delta_{ij}\) and the viscous stress tensor \(\tau_{ij}\).

1. Multiply Equation (1) by \(u_i\) and manipulate the result appropriately to arrive at the following form of the energy equation

\[ \frac{\rho}{2} \frac{Du_i u_i}{Dt} = \rho f_i u_i + \frac{\partial \sigma_{ij}}{\partial x_j} u_i + p \frac{\partial u_i}{\partial x_j} - \tau_{ij} \frac{\partial u_i}{\partial x_j} \]  

(2)

2. Show for a Newtonian fluid that the last term in equation 2 is equal to

\[ \Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} = \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \]  

(3)

which is a positive definite quantity. This quantity always acts to increase irreversibly the internal energy of the fluid at the expense of kinetic energy. Explain why this term is also referred to as the “deformation work.”

3. Give a physical interpretation of the remaining three terms on the right-hand-side of equation 2.
4. If there is no body force and if the incompressible fluid is completely confined in a rigid and fixed vessel, use Gauss’s theorem to show that

\[
\frac{\rho}{2} \frac{\partial}{\partial t} \int_V u_i u_i dV = -\int_V \Phi dV
\]  

(4)

From this equation and the result of step 2 above, prove that the viscosity coefficient \( \mu \) can only be positive.

2 Flow Work in Couette Flow

The energy equation derived above is given by

\[
\frac{\rho}{2} \frac{D u_i u_i}{Dt} = \rho f_i u_i + \frac{\partial \sigma_{ij} u_i}{\partial x_j} + p \frac{\partial u_i}{\partial x_i} - \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2
\]  

(5)

1. Write the energy equation 5 for the plane Couette flow given by \( u_1 = \frac{U}{\pi} x_2 \).
2. Show that \( \tau_{ij} \) is constant for this flow field.
3. Show that the kinetic energy of the flow is also constant.
4. Finally, show that an input stress on the boundary is necessary to maintain this flow and explain what forms of energy the boundary work is exchanged among.

3 Vorticity Equation

The vorticity equation is obtained by taking the curl of the momentum conservation equation. In homework #2, we showed that

\[
\frac{D \vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} q^2 \right) + \vec{\omega} \times \vec{u}
\]  

(6)

Use this result and manipulate the viscous term of the Navier-Stokes equation to show that momentum conservation for a constant-density fluid may also be written as

\[
\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i} \left( \frac{p}{\rho} + \frac{1}{2} u_j u_j \right) + \epsilon_{ijk} u_j \omega_k - \nu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}
\]  

(7)

Take the curl of this equation to show that the vorticity equation reduces to

\[
\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \epsilon_{ij} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}
\]  

(8)

where \( \omega_i \) is the vorticity vector and \( \epsilon_{ij} \) is the rate of strain tensor. In order to obtain the result above, you may find the following hints helpful:

- Recall the tensor identity \( \epsilon_{pqi} \epsilon_{ijk} = \delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj} \).
- The alternating tensor \( \epsilon_{ijk} \) is anti-symmetric, so its product with any symmetric tensors will be zero.
\begin{itemize}
\item The vorticity has zero divergence since the divergence of the curl of a vector is zero.
\item The deformation tensor can be decomposed into a symmetric part $\epsilon_{ij}$ and anti-symmetric part $\Omega_{ij}$.
\item $-\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = \frac{1}{2}\epsilon_{ijk}\omega_j\omega_k$ can only be true if both sides of the equation are zero.
\end{itemize}