OCEN 678
Fluid Dynamics for Ocean and Environmental Engineering
Problems
1 Module 1: Fundamental Equations

1.1 Introduction, Tensor Algebra, and Kinematics
1.1.1 Fluid Mechanics Review

A velocity vector is given by

\[ \vec{u} = (3x + y) \hat{i} + (2x - 3y) \hat{j} + 2xy \exp\left(-\frac{t}{2}\right) \hat{k} \]  

(1)

where \( x \) and \( y \) are Cartesian coordinates and \( t \) is time. Answer the following. For each yes/no question, please explain your answer.

1. Find the unsteady part of the acceleration \( \partial \vec{u} / \partial t \).
2. Find the convective acceleration \( \vec{u} \cdot \nabla \vec{u} \).
3. What is the total acceleration \( D\vec{u} / Dt \)?
4. Is the flow steady? Is the flow uniform?
5. Find the divergence of the flow field \( \partial u_i / \partial x_i \). If this fluid is incompressible, does this flow field satisfy conservation of mass?
6. Find \( \nabla \times \vec{u} \). Is the flow irrotational?
Repeat Problem 1.1.1 using the velocity field given by

\[ \vec{v} = \left( -\frac{Ay}{x^2 + y^2} e^{-kt} \right) \hat{i} + \left( \frac{Ax}{x^2 + y^2} e^{-kt} \right) \hat{j} + Btk \hat{k} \]  

(2)

\( A, B, \) and \( k \) are constants, \( x \) and \( y \) are Cartesian coordinates, and \( t \) is time.

In addition to the calculations above, also plot the velocity vector field in the plane \( z = 0 \) at the time \( t = 0 \) for the case that \( A = 1 \text{ m}^2/\text{s} \).
1.1.3 Velocity Fields

A certain velocity field is given by

\[ \vec{u} = (3x + y)\vec{i} + (2x - 3y)\vec{j} + \frac{1}{x^2 + y^2}\vec{k} \]  

1. If the fluid is incompressible, does this flow field satisfy conservation of mass?

2. Obtain an expression for the \( y \)-component of the acceleration, \( a_y = Dv/ Dt \).

3. Is this flow field steady? Uniform?

4. Compute the vorticity vector \( \nabla \times \vec{u} \) at the point \( \vec{x} = (1, 1, 0) \).

5. Obtain an expression in this velocity field for the shear stress tensor for a Newtonian fluid, given by

\[ \tau_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

6. Compute the kinetic energy per unit mass at the point \( \vec{x} = (2, 1, 1) \) given by \( \text{K.E.} = \frac{1}{2} \vec{u} \cdot \vec{u} \).
1.1.4 Tensor Notation – Operations

Write out the following expressions using tensor notation. Assume all vectors have three components and matrices are \((3 \times 3)\).

\[
\begin{align*}
\frac{\partial \vec{u}}{\partial t} &= \quad \vec{F} = \\
\vec{u} \cdot \nabla c &= \quad \vec{u} \cdot \nabla \vec{u} = \\
\nabla^2 \phi &= \\
\nabla (\nabla \cdot \vec{u}) &= \quad \nabla \times \vec{F} = \\
\nabla \times (\nabla \times \vec{u}) &= \quad \text{trace } \mathbf{A} = \\
\mathbf{A} \vec{u} &= \quad (5)
\end{align*}
\]
1.1.5 Tensor Notation – Conservation Equations

Write the following equations using tensor notation. For vector equations, use $i$ as the free index.

1. Energy Equation:

$$\rho \frac{\partial e}{\partial t} + (\rho \vec{u} \cdot \nabla) e = -p \nabla \cdot \vec{u} + \nabla \cdot (k \nabla T) + \Phi$$  \hspace{1cm} (6)

Here, $\rho$ is the density, $e$ is the internal energy, $\vec{u}$ is the velocity, $p$ is the pressure, $k$ is the scalar thermal conductivity, $T$ is the temperature, and $\Phi$ is the scalar dissipation function.

2. Vorticity Equation:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} + \nu \nabla^2 \vec{\omega}$$ \hspace{1cm} (7)

Variables appearing in more than one of these equations, such as for $\vec{u}$, retain their same definition for each equation. The new variables in the vorticity equation are the vector vorticity $\vec{\omega}$ and the scalar kinematic viscosity $\nu$.

3. Enstrophy Equation

$$\frac{\partial}{\partial t} \left( \frac{\vec{\omega}^2}{2} \right) + (\vec{u} \cdot \nabla) \frac{\vec{\omega}^2}{2} = \vec{\omega}^T S \vec{\omega} - \nu (\nabla \times \vec{\omega})^2 + \nu \nabla \cdot [\vec{\omega} \times (\nabla \times \vec{\omega})]$$ \hspace{1cm} (8)

Enstrophy is defined as $\vec{\omega}^2/2$. This is a scalar equation; in the notation used here, the square of a vector is given by

$$\vec{a}^2 = \vec{a}^T \vec{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \vec{a} \cdot \vec{a}$$ \hspace{1cm} (9)

where $^T$ is the matrix transpose operator. In Equation (8), $S$ is the $(3 \times 3)$ matrix of the rate of strain tensor $e_{ij}$. 
1.1.6 Special Tensors

1. Evaluate the following expressions (find their numerical values)

\[
\delta_{ij}\delta_{ji} = \\
\epsilon_{ijk}\epsilon_{ijk} = (10)
\]

2. Prove the following relationships

\[
\epsilon_{pqj}\epsilon_{pqi} = 2\delta_{ij} \\
\epsilon_{pqj}\epsilon_{sqr} = \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn} (11)
\]

using the identity (given here without proof)

\[
\epsilon_{pqj}\epsilon_{mnr} = \det \begin{bmatrix} 
\delta_{mp} & \delta_{mq} & \delta_{ms} \\
\delta_{np} & \delta_{nq} & \delta_{ns} \\
\delta_{rp} & \delta_{rq} & \delta_{rs} 
\end{bmatrix} (12)
\]
1.1.7 Vector Identities

Expand the following vector identities in tensor notation to demonstrate that they are true.

1. $\nabla \times (\phi \vec{a}) = \nabla \phi \times \vec{a} + \phi (\nabla \times \vec{a})$

2. $\nabla \times \nabla \phi = 0$

3. $\nabla \cdot (\nabla \times \vec{u}) = 0$

4. $\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\nabla \times \vec{u}) - \vec{u} \cdot (\nabla \times \vec{v})$
1.1.8 Symmetric and Antisymmetric Tensors

In this section, consider the two tensors:

\[
\mathbf{C} = \begin{bmatrix}
3 & 4 & 5 \\
-2 & 3 & 4 \\
3 & -5 & 2
\end{bmatrix}
\]

\[
\mathbf{R} = \begin{bmatrix}
1 & -4 & 5 \\
-4 & 3 & 3 \\
5 & 3 & 5
\end{bmatrix}
\] (13)

1. Write the matrix multiplication \( \mathbf{CR} \) and \( \mathbf{RC} \) using indicial notation and implied summations.

2. Compute the numerical result of the matrix multiplications in the previous question.

3. Write the matrix \( \mathbf{C} \) as the sum of a symmetric matrix \( \mathbf{S} \) and an antisymmetric matrix \( \mathbf{A} \).

4. Compute the matrix multiplications \( \mathbf{SR} \) and \( \mathbf{AR} \). What types of matrices are the result? When multiplying by \( \mathbf{R} \), does the order of the matrix multiplication matter? Why do you think this is?

5. Compute the doubly contracted product (two implied sums)

\[
P = R_{ij}C_{ij}
\] (14)

6. Compare the result to the products \( R_{ij}A_{ij} \) and \( R_{ij}S_{ij} \). In general, what is the doubly contracted product of a symmetric and antisymmetric tensor?
1.1.9 Dredge Dynamic Positioning

The hydraulic dredge shown in the figure below is used to dredge sand from a coastal inlet. Estimate the thrust (force in Newtons) needed from the propeller to hold the boat stationary. Assume the specific gravity of the sand/water mixture is $SG = 1.2$.

Figure 1: Schematic of a cutter-suction dredge operation. Taken from Munson et al. (2009).
1.1.10 Forces in Pipe Networks

An orifice meter is a common way to measure flow rate in pipe networks in chemical processing plants. Consider the T-connection and orifice meter in Figure 4. A 1 m$^3$/s flow rate of water enters the large pipe from left to right and the pressure in the system is given by the gauges in the figure. The orifice meter is a flat plate that obstructs the flow with a center ring exit. Assume that 40% of the inflow exits through the free jet.

1. If the inner diameter of the orifice plate is 15 cm, calculate the reaction force necessary to hold the orifice plate in place. Assume the pressure at the plate throat is that given by the 150 kPa gauge. Draw your selected control volume, label the flows and applied forces, and calculate the reaction force in Newtons.

2. Calculate the reaction forces on the flange holding the T-connection in place. Again, draw a new control volume, label the flows and applied forces, and calculate the reaction forces ($x$- and $y$-direction) in Newtons.
1.1.11 Flow Kinematics

The plane vortex flow has the velocity components

\[ u_r = 0, \quad u_\theta = \frac{c}{r} \]  

for \( r > 0 \). Verify that the velocity field is solenoidal and irrotational. Consider a fluid element initially square when it lies on the \( x \)-axis at the point \((r_0, 0)\). The element moves with the flow to the point \((r_0, \theta)\) (refer to Figure 3). What are the angular displacements of the side AB and AC from their initial configurations? What is the average of these angular displacements?

![Figure 3: Schematic of the motion of a fluid element initially at \((r_0, 0)\) to a new position at \((r_0, \theta)\) with angular displacements of side AB and AC.](image-url)
1.1.12 Vector Identities

Use tensor notation to show that the following vector identities are true.

1. \( \nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a} \) \hfill (16)

2. \( (\vec{a} \cdot \nabla)\vec{a} = \frac{1}{2} \nabla(\vec{a} \cdot \vec{a}) - \vec{a} \times (\nabla \times \vec{a}) \) \hfill (17)
1.1.13 Tensor Calculus

A symmetric second-order Cartesian tensor is defined by

\[ T_{ij} = \delta_{ij} - 3x_i x_j. \]  

(18)

Evaluate the following surface integrals, each taken over the surface of the unit sphere:

1. \[ \int T_{ij} dS \]  
2. \[ \int T_{ik} T_{kj} dS \]  
3. \[ \int x_i T_{jk} dS \]  

(19)  
(20)  
(21)
1.1.14 Velocity Fields (15 points)

The velocity field for the flow of an incompressible Newtonian fluid over a porous plate is given by

\[ u_1 = U \left( 1 - e^{-(V/\nu)x_2} \right) \]
\[ u_2 = V \]

where \( U \) and \( V \) are constants, \( \nu \) is the kinematic viscosity, and \( x_2 \) is directed normal to the plate.

1. Show that the flow field is divergence free?

2. Is this flow field irrotational?

3. Obtain an expression for the total acceleration \( Du_i/Dt \).

4. Write all components of the total stress tensor \( \sigma_{ij} \) for this velocity field.

5. Compute the kinetic energy per unit mass given by \( \frac{1}{2} u_i u_i \).
1.2 Strain, Vorticity, and the Reynolds Transport Theorem
1.2.1 Kinematics and Gauss’s Theorem

The velocity field of a certain flow is given by

\[ u_i = (9x_1^2 + 2x_2, 10x_1, -2x_2x_3^2) \] (23)

1. Is this velocity field divergence free?

2. Calculate the vorticity vector for the given flow field at any point \((x_1, x_2)\) on the plane \(x_3 = 5\).

3. Calculate the net flux through the surface bounded by \(x_2 = 1, -1 \leq x_1 \leq 1,\) and \(0 \leq x_3 \leq 1\) with unit normal pointing in the positive \(x_2\)-direction

4. Verify the validity of Gauss’s theorem

\[ \int_V \nabla \cdot \vec{u}dV = \int_S \vec{u} \cdot \hat{n}dA \] (24)

by integrating over the sphere defined by \(x^2 + y^2 + z^2 = a^2\).
1.2.2 Material Derivative

Using indicial notation (and without using any vector identities), demonstrate that the acceleration of a fluid particle is given by

\[
\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} q^2 \right) + \vec{\omega} \times \vec{u}
\] (25)

where \( q = \sqrt{\vec{u} \cdot \vec{u}} \) is the magnitude of the velocity and \( \vec{\omega} \) is the vorticity, given by \( \vec{\omega} = \nabla \times \vec{u} \).
1.2.3 Lagrangian Perspective

Plot the trajectory of a particle released from the point (1,1) at time $t = 0$ for the velocity field given by the parametric equations

$$u = x(1 + t), \quad v = 1$$

(26)
1.2.4 Unit Normal Vectors

In the design of a submarine, it is desired to know the hydrostatic pressure over a surface given the paraboloid \( z = x^2 + y^2 \) under the plane \( z = 4 \). Compute the outward-pointing unit normal to this surface.
1.2.5 Gauss Theorem

The velocity field of a certain flow is given by

\[
\begin{pmatrix}
\frac{5y}{\sqrt{x^2 + y^2}} & \frac{5x}{\sqrt{x^2 + y^2}} & \frac{1}{\sqrt{x^2 + y^2}}
\end{pmatrix}
\]  \hspace{1cm} (27)

1. Is this velocity field divergence free?

2. Verify the validity of Gauss’s theorem

\[
\int_V \nabla \cdot \vec{u} \, dV = \int_S \vec{u} \cdot \hat{n} \, dA \hspace{1cm} (28)
\]

by integrating both sides of the equation separately over the solid box \([-1, 1] \times [-1, 1] \times [-1, 1]\) with outward-pointing normal.

3. Apply Gauss’s theorem to find a surface integral that is equivalent to the following volume integral (do not evaluate the resulting integral)

\[
\int_V \frac{\partial \sigma_{ij} u_i}{\partial x_j} \, dV \hspace{1cm} (29)
\]

where \(\sigma_{ij}\) is an arbitrary second-order tensor.
1.2.6 Vorticity Equation

The conservation of momentum for a fluid of constant density $\rho$ may be written as

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i} \left( p \rho + \frac{1}{2} u_j u_j \right) + \epsilon_{ijk} u_j \omega_k - \nu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$$

(30)

where $u_i$ is the velocity vector, $\omega_i$ is the vorticity vector, $p$ is the pressure, and $\nu$ is the kinematic viscosity.

Take the curl of this equation to obtain the following form of the vorticity equation.

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j e_{ij} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2}$$

(31)

where $e_{ij}$ is the rate of strain tensor. Do not use any vector identities; rather, arrive at this form of the equation by tensor manipulation and algebra.

To obtain this result, you may use the following:

- $-\frac{1}{2} \epsilon_{ijk} \omega_j \omega_k = \frac{1}{2} \epsilon_{ijk} \omega_j \omega_k$ is only true if both sides of the equation are zero.

- The divergence of the vorticity is zero since the divergence of the curl of a vector is zero.
1.2.7 Energy Equation

The conservation of kinetic energy for a Newtonian fluid may be written as

\[
\frac{\rho}{2} \frac{Du_i u_i}{Dt} = \rho g_i u_i + \frac{\partial \sigma_{ij} u_i}{\partial x_j} + p \frac{\partial u_i}{\partial x_i} - \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2
\]  

(32)

with

\[
\sigma_{ij} = -\delta_{ij} p + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]  

(33)

where \( g_i = (0, 0, -g) \) is the gravity vector, and \( \lambda \) and \( \mu \) are viscosity coefficients.

Stokes second problem gives the solution for the flow above an oscillating plate. The plate is assumed to be an infinite plane in \( xz \)-space that oscillates sinusoidally in the \( x \) direction with frequency \( \eta \) and maximum speed \( U \). The velocity field above the plate (\( y \)-direction) is found to be

\[
\frac{u(y, t)}{U} = \exp \left( -\sqrt{\frac{\eta}{2 \nu}} y \right) \cos \left( \eta t - \sqrt{\frac{\eta}{2 \nu}} y \right)
\]  

(34)

where \( \mu / \rho = \nu \).

Evaluate each term in the conservation of kinetic energy equation for this velocity field. First, substitute \( u_1 = f(x_2) \) to determine which terms remain in the equation. Then substitute the given velocity profile to evaluate the magnitude of each non-zero term. Which terms are exactly zero? Which terms a small compared to others (if any)? Based on this comparative scale analysis, write down the simplest form of the conservation of kinetic energy equation that could be used to analyze this flow field.
1.2.8 Vector Identities (10 points)

Use tensor notation to show that the following vector identity is true for any vector \( \vec{u} \) and \( \vec{v} \):

\[
\nabla \times (\vec{u} \times \vec{v}) = (\vec{v} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{v} + \vec{u}(\nabla \cdot \vec{v}) - \vec{v}(\nabla \cdot \vec{u})
\]

(35)
1.2.9 Kinematics

Find the rates of deformation and vorticity in Cartesian coordinates for the velocity field

\[ \mathbf{v} = \left( \frac{Bx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{By}{(x^2 + y^2 + z^2)^{3/2}}, \frac{Bz}{(x^2 + y^2 + z^2)^{3/2}} \right) \]

(36)
1.3 Governing Equations
1.3.1 Governing Equations

1. What is the general expression defining an incompressible fluid? Explain how fluid flow problems in ocean engineering may assume an incompressible fluid that has variable density.

2. The general form of the momentum equation is

\[
\frac{D u_j}{Dt} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j
\]  

(37)

Take the curl of this equation, \( \epsilon_{pqj} \partial(\cdot)_j/\partial x_q \), to obtain the general form of the vorticity equation. Take care to remember that \( \rho \) is not a constant. You may also make use of the relation demonstrated in Homework 2:

\[
\frac{D u_j}{Dt} = \frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{1}{2} u_k u_k \right) + \epsilon_{ijk} \omega_i u_k
\]  

(38)

where \( \omega_i \) is the vorticity, defined as \( \nabla \times \vec{u} \). Work this problem in indicial notation, and do not use any vector identities without proof.

3. Write down the total stress tensor \( \sigma_{ij} \) for a fluid subject to the standard Ocean Engineering assumptions (incompressible, Newtonian fluid with constant viscosity coefficients).

4. Substitute the Ocean Engineering stress tensor into the general vorticity equation to obtain the vorticity equation in Ocean Engineering. Write the resulting equation in vector notation.

Hint: If \( \rho \) is constant, the classical vorticity equation for a Newtonian fluid is

\[
\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{\omega}.
\]  

(39)

You should obtain a similar equation containing extra terms that account for the variable density.
1.3.2 Differentiation

Conservation of mass for an incompressible fluid is given by

\[ \nabla \cdot \vec{u} = 0 \]  \hspace{1cm} (40)

Use this result to simplify each of the following expressions (if possible).

\[
\frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} = \\
\frac{\partial}{\partial x_j} \left( \frac{\partial u_j}{\partial x_i} \right) = \\
\delta_{ij} \frac{\partial u_i}{\partial x_j} = \\
\epsilon_{ijk} \frac{\partial u_j}{\partial x_i} =
\]  \hspace{1cm} (41)
1.3.3 Conservation of Energy

The conservation of energy equation results from an application of the first law of thermodynamics to a fluid element; namely, the change in internal energy due to an event is equal to the sum of the work done on the system during the event and any heat that was added. Application of this law results in:

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_j}{\partial x_i} - \frac{\partial q_j}{\partial x_j}$$  \hspace{1cm} (42)

where $e$ is the internal energy per unit mass and $q_j$ is the rate of heat flux vector per unit area.

The constitutive equation for heat transfer in a Newtonian fluid is given by Fourier’s law of heat conduction:

$$q_i = -k \frac{\partial T}{\partial x_i}$$  \hspace{1cm} (43)

where $k$ is the thermal conductivity and $T$ is the temperature. Substitute this relation along with the standard assumptions of ocean engineering (incompressible, Newtonian fluid with constant, molecular viscosity and heat conductivity coefficients) to obtain a simplified version of the energy equation. When would this equation be mathematically coupled to the Navier-Stokes equations? For what types of flows in ocean engineering might this occur?
1.3.4 Kinetic energy

The momentum equation for an incompressible fluid in indicial notation can be given by

\[ \rho \frac{D u_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i \]  

(44)

where \( \sigma_{ij} = -p \delta_{ij} + \tau_{ij} \) represents the sum of the pressure tensor \(-p \delta_{ij}\) and the viscous stress tensor \(\tau_{ij}\).

1. Multiply Equation (44) by \(u_i\) and manipulate the result appropriately to arrive at the following form of the energy equation

\[ \frac{\rho}{2} \frac{D u_i u_i}{Dt} = \rho f_i u_i + \frac{\partial \sigma_{ij} u_i}{\partial x_j} + \frac{p}{\rho} \frac{\partial u_i}{\partial x_i} - \tau_{ij} \frac{\partial u_i}{\partial x_j} \]  

(45)

2. Show for a Newtonian fluid that the last term in equation 45 is equal to

\[ \Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \]  

(46)

which is a positive definite quantity. This quantity always acts to increase irreversibly the internal energy of the fluid at the expense of kinetic energy. Explain why this term is also referred to as the “deformation work.”

3. Give a physical interpretation of the remaining three terms on the right-hand-side of Equation (45).

4. If there is no body force and if the incompressible fluid is completely confined in a rigid and fixed vessel, use Gauss’s theorem to show that

\[ \frac{\rho}{2} \frac{\partial}{\partial t} \int_V u_i u_i dV = - \int_V \Phi dV \]  

(47)

From this equation and the result of step 2 above, prove that the viscosity coefficient \(\mu\) can only be positive.
The general solution for Couette flow between two infinite, parallel plates with a non-zero, but constant, pressure gradient \( dp/dx \) is given by

\[
u_i = \left( \frac{1}{2\mu} \frac{dp}{dx_1} x_2 (x_2 - h) + \frac{Ux_2}{h}, 0, 0 \right)
\]

where \( U \) is the velocity of the top plate in the \( x_1 \)-direction and \( h \) is the distance between the plates in the \( x_2 \)-direction; the bottom plate is stationary and located at \( x_2 = 0 \).

1. Substitute this velocity profile into the general equation for the stress tensor for a Newtonian fluid, given by

\[
\sigma_{ij} = -\delta_{ij} p + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

(49)

to find all nine terms of the stress tensor.

2. Using this result, compute the shear stress tensor \( \tau_{ij} \) on the surface \( x_2 = 0 \), using the values \( h = 0.25 \text{ m}, dp/dx_1 = -0.1 \text{ Pa/m}, \text{ and } U = 1 \text{ m/s} \). The fluid is water, for which the dynamic viscosity is \( 1 \cdot 10^{-3} \text{ Ns/m}^2 \). Recall that \( \sigma_{ij} = -\delta_{ij} p + \tau_{ij} \).

3. Substitute Equation (65) into the following form of the Navier-Stokes equations to determine all non-zero terms

\[
\begin{align*}
\frac{\partial u_i}{\partial x_i} &= 0 \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}
\end{align*}
\]

(50)

Since there is no body force in the momentum equation, what kind of pressure must \( p \) represent? What does this also tell you about the density of the fluid?
1.3.6 Flow Work in Stokes’ First Problem

The energy equation derived in the last problem is given by

\[
\frac{\rho}{2} \frac{Du_i u_i}{Dt} = \rho f_i u_i + \frac{\partial \sigma_{ij} u_i}{\partial x_j} + p \frac{\partial u_i}{\partial x_i} - \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \tag{51}
\]

For the following questions, use the solution to Stokes’ first problem (the velocity above an impulsively started plate moving at constant velocity \(U\)) given by

\[
u(y, t) = U \left( 1 - \text{erf} \left( \frac{y}{2 \sqrt{\nu t}} \right) \right) \tag{52}
\]

1. Substitute this velocity profile into Equation (51) to obtain the energy equation for this flow field.

2. Calculate \(\tau_{ij}\) for this flow field. Is it constant? If not, on what does it depend?

3. Calculate the kinetic energy of the flow field. Is it constant? If not, on what does it depend?

4. Finally, show that an input stress on the boundary \((y = 0)\) is necessary to maintain this flow and explain what forms of energy the boundary work is exchanged among.
1.3.7 Vorticity Equation

The vorticity equation is obtained by taking the curl of the momentum conservation equation. In homework #2, we showed that

\[ \frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} q^2 \right) + \vec{\omega} \times \vec{u} \] \hfill (53)

Use this result and manipulate the viscous term of the Navier-Stokes equation to show that momentum conservation for a constant-density fluid may also be written as

\[ \frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i} \left( \frac{p}{\rho} + \frac{1}{2} u_j u_j \right) + \epsilon_{ijk} u_j \omega_k - \nu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j} \] \hfill (54)

Take the curl of this equation to show that the vorticity equation reduces to

\[ \frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \epsilon_{ij} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2} \] \hfill (55)

where \( \omega_i \) is the vorticity vector and \( \epsilon_{ij} \) is the rate of strain tensor. In order to obtain the result above, you may find the following hints helpful:

- Recall the tensor identity \( \epsilon_{pqi} \epsilon_{ijk} = \delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj} \).
- The alternating tensor \( \epsilon_{ijk} \) is anti-symmetric, so its product with any symmetric tensors will be zero.
- The vorticity has zero divergence since the divergence of the curl of a vector is zero.
- The deformation tensor can be decomposed into a symmetric part \( \epsilon_{ij} \) and anti-symmetric part \( \Omega_{ij} \).
- \( -\frac{1}{2} \epsilon_{ijk} \omega_j \omega_k = \frac{1}{2} \epsilon_{ijk} \omega_j \omega_k \) can only be true if both sides of the equation are zero.
1.3.8 Assumptions

List all necessary assumptions to arrive at the following form of the Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial u_i}{\partial x_i} &= 0 \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_i}{\partial x_j^2}
\end{align*}
\]  

(56)

Would any of the following situations violate these assumptions, why or why not?

- Solution for the flow field around an anchor as it falls through a constant-density fluid far from the free surface.
  - Solution for the flow field around mooring lines for an offshore platform.
  - Solution for the wake behind a submerged submarine.
  - Solution for the internal waves in a stratified alpine lake.
  - Solution for the shock wave behind a supersonic jet aircraft.
  - Solution for a mud flow modeled as a Bingham plastic (substance that does not move until a critical non-zero shear stress is applied).
  - Solution for drag reduction due to polymer addition in the boundary layer of a ship, where the polymers give an anisotropic stress tensor.
  - Solution for the flow field around nanoparticles of length $10^{-9}$ m.
1.3.9 Viscosity Coefficient

For an incompressible fluid confined in a rigid and fixed container where there is no body force, the energy equation can be summed over the volume of fluid in the container to obtain

\[
\frac{\rho}{2} \frac{\partial}{\partial t} \int_V u_iu_idV = - \int_V \tau_{ij} \frac{\partial u_i}{\partial x_j} dV \tag{57}
\]

where \( \tau_{ij} \) is the dynamic portion of the stress tensor \( \sigma_{ij} \). Substitute \( \tau_{ij} \) for a Newtonian fluid and show that the integrand on the right-hand-side of the above equation becomes

\[
\frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \tag{58}
\]

which is \( \mu \) times a positive definite quantity. Use this result and the physical interpretation of the energy equation to argue that \( \mu \) must be greater than zero.
1.3.10 Mud Flow Dynamics

Submarine landslides often result in the flow of muds, which are non-Newtonian fluids that behave similarly to Bingham fluids. The Bingham model states that the stress strain relationship is linear, but that a non-zero yield stress must be exceeded before there is fluid motion. To write down the stress-strain relationships for a generalized, three-dimensional flow, we must introduce two new quantities:

\[
\Gamma_T = \frac{1}{2} \left( \tau_{ij} \tau_{ij} - (\tau_{kk})^2 \right)
\]
\[
\Gamma_E = \frac{1}{2} \left( e_{ij} e_{ij} - (e_{kk})^2 \right).
\]

The Bingham plastic law is then

\[
e_{ij} = 0, \quad \text{if } \sqrt{\Gamma_T} < \tau_c
\]
\[
\tau_{ij} = 2\mu e_{ij} + \tau_c \frac{e_{ij}}{\sqrt{\Gamma_E}}, \quad \text{if } \sqrt{\Gamma_T} \geq \tau_c
\]

where \( \tau_c \) is the scalar yield stress.

1. Write the Bingham plastic law in simple, parallel shear flow, where \( u = f(y) \).

2. Obtain an expression for the three-dimensional momentum equation for the moving, incompressible fluid, where \( \sqrt{\Gamma_T} \geq \tau_c \). Write each part of the equation in terms of the velocity vector and not simply \( e_{ij} \) or \( \tau_{ij} \).
1.3.11 Poisson Pressure Equation (10 points)

Take the divergence of the momentum equation for a Newtonian fluid of constant density to obtain a new equation. This equation is known as the Poisson pressure equation and relates the pressure to the velocity field.
1.3.12 Manipulation of the Momentum Equation

The momentum equation for an incompressible Newtonian fluid in indicial notation can be given by

\[ \rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho f_i + \mu \frac{\partial^2 u_i}{\partial x_j^2} \]  

where \( u_i \) is the velocity vector, \( p \) is the pressure, \( f_i \) is the gravity vector \((0, 0, -g)\), and \( \mu \) is a constant dynamic viscosity coefficient.

1. Take the dot product of \( u_i \) with the above equation to obtain an expression for the conservation of kinetic energy \( \frac{1}{2} u_i^2 \). Give the physical interpretation of the terms \( p \frac{\partial u_i}{\partial x_i} \), \( \mu \frac{\partial^2 (\frac{1}{2} u_i^2)}{\partial x_j^2} \), and \( \frac{\partial u_i}{\partial x_j} \) in the resulting equation. For an incompressible fluid, which of these terms, if any, do you expect to be zero? Why or why not? For the term containing \( \frac{\partial u_i}{\partial x_j} \), what effect does this term have on the kinetic energy \( \frac{1}{2} u_i^2 \) as we follow the fluid motion, and what physical process does this represent?

2. Take the divergence of the above equation for a fluid of constant density to obtain the Poisson pressure equation that relates the pressure to the velocity field. For an incompressible fluid, which of the terms in this equation should be zero?
1.3.13 Dissipation Function

In the energy equation, the term $\sigma_{ij}(\partial u_j / \partial u_i)$ represents the work done by surface forces. Write out this expression in terms of velocity and pressure for a Newtonian fluid. The terms multiplying the viscosity coefficients $\lambda$ and $\mu$ are collectively called the dissipation function $\Phi$ and represent irreversible energy losses. Show for an incompressible Newtonian fluid that

$$\Phi = \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2$$

(62)
1.3.14 Cylindrical Coordinates

Obtain the full continuity equation (compressible version) in cylindrical coordinates by expanding its vector form in Cartesian coordinates using the following relationships between the Cartesian system $\vec{x} = (x, y, z)$ and $\vec{u} = (u, v, w)$ and cylindrical coordinate system $\vec{r} = (R, \theta, z)$ and $\vec{u} = (u_R, u_\theta, u_z)$:

$$R = \sqrt{x^2 + y^2}$$
$$\theta = \arctan(y/x)$$
$$x = R \cos \theta$$
$$y = R \sin \theta$$
$$z = z$$
$$u = u_R \cos \theta - u_\theta \sin \theta$$
$$v = u_R \sin \theta + u_\theta \cos \theta$$
$$w = u_z$$

Write down the corresponding momentum equations in cylindrical coordinates using a textbook reference. Expand all $\nabla$ and $\nabla^2$ terms so that everything is expressed in terms of partial derivatives.

Also write down an expression for the vorticity in cylindrical coordinates in terms of partial derivatives and unit vectors.
2 Module 2: Viscous Flows

2.1 Exact and Approximate Solutions
2.1.1 Poiseuille Flow

The laminar flow through a pipe is called Poiseuille Flow. Derive equations for the velocity profile \( u(r) \) and volume flux \( Q \) for steady pipe flow using cylindrical-polar coordinates (refer to Figure 4). Let the radius of the pipe be \( R \), let \( z \) be directed down the pipe axis, and let the velocity vector be given by \( \vec{u} = (u_r, u_\theta, u_z) \). Assume the pipe is infinitely long, the flow is at steady state, and there is no flow in the \( \theta \)-direction. In particular, do the following:

1. State the boundary conditions.

2. State all simplifications with their justifications (e.g. \( \partial \vec{u} / \partial \theta = 0 \) by symmetry) that can be deduced from the given problem statement and geometry.

3. Show using conservation of mass and momentum, that the above conditions reduce to the following problem:

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{du_z}{dr} \right) = \frac{1}{\mu} \frac{dp}{dz}
\]

\( u_z(R) = 0 \)

\( u_z(0) = \text{finite} \)

\( \frac{dp}{dz} = \text{constant} \) \hspace{1cm} (64)

4. Solve these equations to obtain the velocity profile \( u_z(r) \).

5. Integrate the velocity profile to obtain an expression for the total volume flux \( Q \).

6. Obtain an expression for the maximum velocity in the pipe. Where does the maximum velocity occur?

7. Plot the velocity profile and compute the volume flow rate and maximum velocity for water for \( R = 0.1 \) m and having a Reynolds number \( Re = UR/\nu \) of 1,500, where \( U \) is the average velocity over the cross section. What is the value of \( dp/dz \) in Pa/m necessary to generate this flow? What is the shear stress at the pipe wall in Pa and what is the shear force per unit length of pipe in N/m?

![Figure 4: Cylindrical-polar coordinate system for Poiseuille flow problem.](image)
2.1.2 Exact Solutions

The flow induced in a deep channel subject to a constant wind forcing can be approximated by
the idealized flow problem depicted in Figure 5. The two side walls at $y = \pm L/2$ are stationary
and impermeable. The top wall at $z = 0$ is impermeable and moving at a constant velocity $U$ in
the positive $x$-direction (into the page). The flow extends to $z = \infty$ in the $z$-direction and $x = \pm \infty$
in the $x$-direction. Assume there is no pressure gradient in the $x$-direction, the fluid is incompressible,
and the fluid is stationary except for the motion that is induced by the moving surface.

1. Dynamic Pressure.

State the assumptions necessary to remove the body force and hydrostatic pressure from the
governing equations so that we solve only for the dynamic part of the pressure term. Write
the Navier-Stokes equations (mass and momentum conservation) under these conditions and
define the variables. Is this set of equations an appropriate set to solve the problem stated
above. Why or why not?

2. Modified Stokes 1st Problem.

If $L = \infty$, then this problem reduces to the steady solution to Stokes 1st problem. Show that
in this case the governing equations reduce to

$$\n \nu \frac{d^2 u}{dz^2} = 0 \quad (65)$$

subject to the boundary conditions

$$u(x, y, 0) = U$$
$$u(x, y, \infty) = U \quad (66)$$

Figure 5: Idealized flow field for surface-stress driven flow between two side walls.
Explain the boundary condition at $z = \infty$. Solve this equation and compare its solution to the unsteady solution to Stokes 1st problem given by

$$\frac{u}{U} = 1 - \text{erf} \left( \frac{y}{2\sqrt{\nu t}} \right)$$

in the limit $t \to \infty$.


Beginning with the Navier-Stokes equations (mass and momentum conservation), derive the governing equation for the exact solution to this flow field for $L$ positive and finite. Be sure to include all simplifications stated in the problem and those that can be deduced from each of the governing equations and boundary conditions. Clearly state any other assumptions you are making, if applicable.

Summarize your final solution by providing a complete list of assumptions and deductions you used to arrive at your governing equation. Where might you look to find solutions to this problem?
2.1.3 Scale Analysis

Consider the thin, unsteady flow of water down an incline as in the case of plane flow down a levee. Assume the levee is long and the flow is uniform along the levee so that a two-dimensional approximation is valid.

1. Use scale analysis to determine the conditions under which all of the acceleration terms may be neglected. What is the required characteristic time scale? What is the allowable Reynolds number $Re_h = U h / \nu$? How is the depth Reynolds number related to the Froude number $Fr = U / \sqrt{gh}$ (hint: compare the viscous and gravity terms in the $x$-momentum equation)?

2. From the above analysis, show that the approximate equations in this case are

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
0 &= \nu \frac{\partial^2 u}{\partial y^2} + g \sin \theta \\
0 &= \frac{1}{\rho} \frac{\partial p}{\partial y} + g \cos \theta
\end{align*}
\]

(68)

3. Solve these equations for the water depth $h$ as a function of volume flow rate $q$ per unit length of levee. Use the no-slip boundary condition at the wall and the stress-free boundary condition at the free surface.

4. For a levee 3 meters high with side slope $\theta = 5^\circ$, determine the greatest flow rate $q$ that satisfies the conditions for the above equations to be valid. What must the time-scale of the variations in the flow rate be so that the flow may be considered steady as we did above? Do you think this model is valid for flow of rain water during drizzle? What about for steady or wave-driven overtopping during a flood? As the Reynolds number increases, which terms must be added to our governing equations?
2.1.4 Liquid Water Flow along an Icicle

Short et al. (Short, Baygents, & Goldstein, “A free-boundary theory for the shape of the ideal dripping icicle,” Physics of Fluids 18, 083101, 2006) analyze the flow of water down an icicle in order to obtain a non-dimensional relationship for the geometry of icicles. The following paragraph is taken from their paper with a slight change in the names of the variables so that it is more consistent with our notation:

\begin{equation}
R_{0}(z) \quad h(z) \quad \gamma
\end{equation}

Figure 6: Schematic of the sheet flow down an icicle far from the tip.

We first consider the water layer flowing down the surface of a growing icicle to set some initial scales. The volumetric flow rate \( Q \) over icicles is typically on the order of tens of milliliters per hour (\( \approx 0.01 \text{ cm}^3/\text{s} \)), and icicle radii are usually in the range of 1–10 cm. To understand the essential features of the flow, consider a cylindrical icicle of radius \( R_0 \), over the surface of which flows an aqueous film of thickness \( h \) (refer to Figure 6(a)). Since \( h \ll R_0 \) over nearly the entire icicle surface, the velocity profile in the layer may be determined as that flowing on a flat surface. Furthermore, we expect the Reynolds number to be low enough that the Stokes approximation is valid \([e.g., Re \sim O(1) \text{ or less}].\) If \( y \) is a coordinate normal to the surface and \( \gamma \) is the angle that the icicle surface makes with respect to the horizontal (see Figure 6(b)), then the Stokes equation for gravity-driven flow is \( \nu d^2u/dy^2 = g \sin \gamma \), where \( g \) is the gravitational acceleration and \( \nu = 0.01 \text{ cm}^2/\text{s} \) is the kinematic viscosity of water. Enforcing no-slip and stress-free boundary conditions at the solid-liquid and liquid-air interfaces, the thickness is

\begin{equation}
h = \left( \frac{3Q\nu}{2\pi g R_0 \sin \gamma} \right)^{1/3}.
\end{equation}

Using typical flow rates and radii, we deduce a layer thickness that is tens of microns and surface velocities \( u_s \approx (gh^2/2\nu) \sin \gamma \) below several mm/s, consistent with known values, yielding \( Re = 0.01 - 0.1 \), well in the laminar regime as anticipated. At distances from the icicle tip comparable to the capillary length (several millimeters), the complex physics of pendant drop detachment takes over and the thickness law Equation (69) ceases to hold.
Verify their analysis by working through the following steps.

1. Write the exact equations for steady, gravity-driven, sheet flow down a cylinder using cylindrical coordinates. Simplify the equations for steady flow and the given geometry. Gravity is the only driving force; for now, leave this body force as \( \vec{f} = (f_R, 0, f_z) \).

2. Use the assumption of shallow sheet flow \( (h \ll R_0) \) to show that the equations from step one can be written in the form for two-dimensional flow over a flat plate. That is, show that the exact equations may be approximated by

\[
\frac{\partial u_R}{\partial R} + \frac{\partial u_z}{\partial z} = 0 \tag{70}
\]

\[
u \frac{\partial u_R}{\partial R} + u_R \frac{\partial u_R}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial R} + \nu \left( \frac{\partial^2 u_R}{\partial R^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + f_R \tag{71}
\]

\[
u \frac{\partial u_z}{\partial R} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 u_z}{\partial R^2} + \frac{\partial^2 u_z}{\partial z^2} \right) + f_z \tag{72}
\]

3. Use the coordinate system in Figure 6(b) to convert the equations in step two to a set of equations in the Cartesian coordinate system \( x - y \) for the flow over a flat plate. Substitute the correct form of the gravity terms based on the geometry in the figure.

4. Use scale analysis and the assumptions of shallow flow \( (h \ll L) \), Stokes approximation \( (Re_h \leq 1) \), and a shallow slope \( (\frac{\pi}{2} - \gamma) \ll 1 \) to obtain the following set of approximate governing equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{73}
\]

\[
\frac{1}{\rho} \frac{\partial p}{\partial y} = g \cos \gamma \tag{74}
\]

\[
\frac{\nu \partial^2 u}{\partial y^2} = g \sin \gamma \tag{75}
\]

5. Solve these simplified equations for the \( u \)-component velocity to obtain

\[
u \frac{\partial u}{\partial y} = g \sin \gamma \left( \frac{y^2}{2} - hy \right) \tag{76}
\]

by applying a no slip boundary condition at the icicle wall \( u(0) = 0 \) and a shallow-flow stress-free boundary condition at the free surface \( (du(h)/dy = 0) \).

6. To obtain an equation for the net flux \( Q \) over the icicle under the assumption \( h \ll R_0 \), we may use the solution for flow over a flat plate just obtained as follows

\[
Q(x) = 2\pi R_0(x) \int_0^h u dy \tag{77}
\]

Solve this equation for the thickness \( h \) to obtain the expression in Equation (69).
7. From real icicles, we measure the net flux to be $Q \approx 0.01 \text{ cm}^3/\text{s}$, a typical radius to be $R_0 \approx 1 - 10 \text{ cm}$, and a typical wall angle as $\gamma \approx 5^\circ$. Using these data, verify the assumptions in this derivation by showing that $h \ll R_0$ and $Re_h \leq 1$. 
2.1.5 Simple Density Currents

Submarine pipelines and well fields are susceptible to damage by submarine land slides and density currents that can extend large distances and shape the topography of the deep oceans. As a simple model for the steady flow in a density current behind the leading edge of the flow, consider the following problem.

Two fluids of different densities and viscosities are flowing down a plane with slope $\theta$ (refer to Figure 7). The flow is unbounded in the lateral direction and assumed steady and infinite in the along-slope direction. The lower layer of fluid occupies the region $0 \leq y \leq a$ and has density $\rho_1$ and viscosity $\mu_1$, while the upper fluid occupies the region $a \leq y \leq b$ and has density $\rho_2$ and viscosity $\mu_2$; $y$ is directed normal to the plane. The lower fluid has the higher density. The surface at $y = b$ is open to the atmosphere so that the boundary condition is that the shear stress is zero; the boundary condition at $y = a$, the interface between the two fluids, is that the shear stress is continuous (i.e., the shear stress at the top of fluid 1 matches the shear stress at the bottom of fluid 2) and no-slip.

Find the pressure and velocity distributions. Plot the velocity distribution for the case of a flow of $1 \text{ m}^3/\text{s/m}$ in each layer, $\theta = 5^\circ$, $\mu_1 = \mu_2 = 1 \cdot 10^3 \text{ kg/m/s}$, and $\rho_1 = 1025$ and $\rho_2 = 1000 \text{ kg/m}^3$. 

Figure 7: Schematic and coordinate system for density current problem.
2.1.6 Vorticity in Couette Flow

Start with the three-dimensional vorticity equation and the simplifications and assumptions of plane Couette flow with a non-zero pressure gradient \( \frac{dp}{dx} \) to derive the effective vorticity equation in Couette flow. Solve this equation to show that the vorticity distribution in the channel must be linear. Use the solution to the velocity profile that we obtained in class to evaluate the integration constants in your equation for vorticity. Why is it difficult to obtain the boundary conditions to the vorticity equation directly without knowledge of the velocity profile?
2.1.7 Wind-driven flow

As a simplified model of wind-driven flow, consider the flow of water in a laterally-infinite domain with a constant shear stress $\tau_h$ applied at the free surface at $y = h$. The wind blows in the $x$-direction, and the reservoir bottom is flat at $y = 0$ (refer to Figure 8). Consider the possibility of a non-zero pressure gradient in the $x$-direction as, for example, due to a tide, but treat the pressure gradient to be uniform and steady. The flow is steady and the applied free-surface shear stress is uniform.

![Figure 8: Schematic of a constant wind blowing on an infinite reservoir.](image)

1. State the simplifications implied by the above problem statement and write down the simplified governing equations for mass and momentum conservation with the appropriate boundary conditions.

2. Solve the above equations to obtain a solution for the velocity profile $u(y)$ in terms of the pressure gradient $dp/dx$ and the applied shear stress $\tau_h$.

3. Apply similar assumptions and simplifications to the following complete form of the energy equation to obtain a simplified energy equation for this flow field:

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_i}{\partial x_j} + k \frac{\partial^2 e}{\partial x^2}$$

where $e$ is the kinematic internal energy given by $c_p T$; $T$ is the temperature, $c_p$ is the specific heat at constant pressure, $\sigma_{ij}$ is the stress tensor and $k$ is the thermal diffusivity. It is acceptable to assume that $T$ is uniform in the $x$-direction. If heating leads to unstable stratification, how would you have to change the solutions to these two problems (that of the velocity profile and heat distribution)?

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2.2 Boundary Layers
2.2.1 Squeeze Film Flow

A wave slamming against the side of a ship is sometimes approximated by a flat plate falling onto the surface of a stagnant water body. If the water is shallow, the flow field under the plate just after it becomes submerged is given by the classical squeeze film flow solution.

\[ y h(t) V_0 L = 0, y = 0 \]

**Figure 9: Schematic of the squeeze film flow.**

The gap shown in the figure has the length \( L \), the height \( h(t) \), and is filled with a fluid of constant density. The top wall of the gap moves downward with the velocity \( V_0 \). The velocity distribution at the exit is

\[ u(y) = 4U_0 \left[ \frac{y}{h(t)} - \left( \frac{y}{h(t)} \right)^2 \right] \tag{79} \]

1. With the initial condition \( h(t = 0) = h_0 \), find a function for the gap height \( h(t) \) as a function of \( V_0, h_0, \) and \( t \).

2. Calculate the maximum velocity \( U_0 \) at the exit as a function of \( L, V_0, h_0, \) and \( t \). To do this, use the conservation of mass and apply it over a control volume containing all of the fluid under the plate.

3. Based on your solution to part 2, what do you argue would be the force on the plate necessary to maintain the constant velocity \( V_0 \) when the plate approaches the ground (\( h(t) \to 0 \)).
2.2.2 Boundary Layer Thickness

An approximate solution for the flat-plate boundary layer is

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

(80)

where $\delta = \sqrt{30}\sqrt{\nu x/U}$. Calculate the displacement and momentum thicknesses as functions of $x$ and compare their values to the boundary layer thickness $\delta$. 
2.2.3 Blasius Boundary Layer Solution

In the lecture, we showed that the equations for the flat-plate boundary layer below a uniform current \( U \) can be written as the similarity solution

\[
f''' + \frac{f}{2}f'' = 0 \tag{81}
\]

where the prime denotes differentiation with respect to the similarity variable \( \eta = y\sqrt{\nu x/U} \), subject to the boundary conditions:

\[
\begin{align*}
&f(0) = 0 \\
&f'(0) = 0 \\
&f'(\infty) = 1 \tag{82}
\end{align*}
\]

Write a computer program to solve this equation using the fourth-order Runge-Kutta method and the programming language of your choice. You may use a built-in solver as long as it is at least 4th-order accurate. Be sure to comment your program and include a listing of the code with your assignment. Do not share blocks of your code with others!

From your numerical solution, also do the following:

1. Plot the velocity profiles \( u(\eta)/U \) and \( v(\eta)/V \) versus \( \eta \) over a large enough range of \( \eta \) to capture the whole velocity profile.

2. At what value of \( \eta \) is \( u/U \) equal to 0.95?

3. At the top of the boundary layer (\( u/U = 0.99 \), or \( \eta = 5.0 \)), what is the value of the non-dimensional vertical velocity \( v/V \)?

4. Numerically integrate your solution of \( u/U \) to obtain the relations for the displacement and momentum thicknesses:

\[
\begin{align*}
\frac{\delta^*}{x} &= \frac{\alpha}{\sqrt{Re_x}} \\
\frac{\theta}{x} &= \frac{\beta}{\sqrt{Re_x}} \tag{83}
\end{align*}
\]

where \( Re_x = Ux/\nu \). Report the values of \( \alpha \) and \( \beta \) to 4 significant figures.

5. Obtain an expression for \( \partial u/\partial y \) at \( y = 0 \). What value does it approach as \( x \to 0 \) and \( x \to \infty \)?

For the case \( x \to 0 \), how do you explain the fact that the real velocity gradient is finite at the tip of a plate?
2.2.4 Friction Drag on a Boat

The flat-plate boundary layer transitions from laminar to turbulent at about a Reynolds number of $Re_x = 3 \cdot 10^5$. For a sail boat cruising at 7 knots, what is the longest beam of boat that would have a fully laminar boundary layer? What is the boundary layer thickness at the stern of the boat? If the wetted width is $1/3$ the length, what is the friction drag force on the boat?
2.2.5 Boundary Layer Velocity Profiles

Use von Karman’s Momentum Integral to fit the velocity profile

\[ u(x, y) = U(x) \left( 1 - e^{-k(x)y} \right) \tag{84} \]

by finding the functional relationship for \( k(x) \) as a function of the plate Reynolds number \( Re_x \) for the Blasius boundary layer with \( U = \text{constant} \). Plot the resulting velocity profile with the Blasius boundary layer solution obtained in the previous assignment. Comment on their similarities and differences.

From your fitted equation, also obtain solutions for \( \tau_0/(\rho U^2/2) \) and \( C_D \). What is the relative error using this exponential function compared to the exact solution obtained from the Blasius solution? Do you think this is an acceptable approximation to the boundary layer solution?
2.2.6 Momentum Integral

The course notes present the use of a sin-function to represent the velocity profile in a boundary layer, resulting in the approximation:

\[ u(x, y) = U(x) \sin \left( \frac{\pi y}{2\delta(x)} \right) \]  \hspace{1cm} (85)

where \( U(x) \) is the velocity outside the boundary layer, \( y \) is the coordinate normal to the surface, and \( \delta(x) \) is the boundary layer thickness.

The velocity field near the surface of a cylinder in slow approach flow is given by

\[ u_\theta = -2U \sin \theta \]  \hspace{1cm} (86)

where \( U \) is the free stream velocity in the \( x \)-direction far from the sphere and \( \theta \) is the polar angle from the \( x \) axis.

Use the von Karman momentum integral to find a model for the boundary layer on a cylinder using the sin-function velocity profile. Solve for the boundary layer thickness and the shear stress on the surface of the cylinder. Integrate the shear stress around the cylinder to obtain a value for the frictional drag coefficient of the cylinder. Compare this value to that in standard fluid mechanics textbooks. How does the solution perform? Discuss your results.
2.2.7 Boundary Layers Simulation of Oil Slicks

Oil spills result in a spreading surface oil slick. When this occurs, the motion of the oil film and the water beneath it is coupled. The physics involves surface tension, viscosity, and nonlinearity, and is very complicated. To obtain a partial understanding of the water motion induced by the spreading oil film, consider the following idealized model.

Let the sea surface at $z = 0$ be initially flat and at rest. At $t = 0$ two plastic sheets descend and are pulled away from $x = 0$ to $x \to \pm\infty$ at the constant speed $U$ along the sea surface. Boundary layers are developed in the water below the sheets, and the sheets are replenished at $x = 0$ throughout the process so that the sea surface is always covered everywhere with the plastic sheets modeling the oil film.

1. Set up the governing equations for unsteady boundary layers (you do not need to do the scale analysis, just present the correct equations).

2. State the initial and boundary conditions.

3. Integrate the momentum equation across the boundary layer to get a depth-integrated momentum equation (the unsteady von Kármán’s momentum integral equation).

4. Assume a parabolic profile for the boundary layer flow to find a partial differential equation for the boundary layer thickness $\delta(x, t)$.

5. Extra credit (5 points): Solve the hyperbolic differential equation for $\delta$. Plot and discuss the results.
2.2.8 Momentum Integral

Generation of wind waves depends on the wind speed and the time over which the wind acts on the waves, often given by the fetch. A simple model for the wind speed over a tidal estuary is

\[ U(x) = 25 - 2.4x^2 \]  

(87)

where \( x \) is in kilometers over the region \(-2.5 \leq x \leq 2.5\), and \( U \) is in m/s. The wind speed starts off low at \( x = -2.5 \) km due to topographic shielding, increases to a maximum in the middle of the estuary, and then decreases due to blockage by topographic features at \( x = 2.5 \) km.

The course notes present the use of a sin-function to represent the velocity profile in a boundary layer, resulting in the approximation:

\[ u(x, y) = U(x) \sin \left( \frac{\pi y}{2\delta(x)} \right) \]  

(88)

where \( y \) is the coordinate normal to the surface and \( \delta(x) \) is the boundary layer thickness.

Apply the von Karman momentum integral with the velocity profile in Equation 87 to find a model of the boundary layer on the lake surface. Plot \( \delta(x) \) from \( x = -2.5 \) km to 2.5 km. Where does the maximum shear stress occur and what is its value? Integrate the shear stress to find the total drag on the estuary surface per unit width. For these calculations, use both a molecular viscosity coefficient of \( 1.5 \cdot 10^{-5} \) m\(^2\)/s and a turbulent viscosity of \( 10^{-1} \) m\(^2\)/s. Which result agrees more closely with your experience.
2.2.9 Boundary Layer Vorticity

Blasius boundary layer solution results from the numerical solution of

\[ \frac{f}{2} \frac{d^2 f}{d\eta^2} + \frac{d^3 f}{d\eta^3} = 0, \tag{89} \]

which is obtained by solving the system of equations

\[
\begin{align*}
\frac{df}{d\eta} &= F \\
\frac{dF}{d\eta} &= H \\
\frac{dH}{d\eta} &= -\frac{f}{2}H 
\end{align*}
\]

subject to the initial conditions

\[
\begin{align*}
f(0) &= 0 \\
F(0) &= 0 \\
F(\infty) &= 1. \tag{91}
\end{align*}
\]

Here, \( \eta \) is the similarity variable \( y/\sqrt{\nu x/U} \), and the velocity profiles are given by \( u = UF(\eta) \) and \( v = \sqrt{U\nu/\bar{x}G(\eta)} \) with \( G = 1/2(\eta df/d\eta - f) \).

This can be solved very simply in Matlab using the shooting method and a 4th-order ODE solver. The main program code is

```matlab
ti = 0.0;
tf = 6.0;
ic = [0.0, 0.0, 0.33206];
[t, y] = ode45('blasius', [ti tf], ic);
```

which calls the program `blasius.m` to evaluate the derivates, given by

```matlab
function [fp] = blasius(t, y)
fp = [y(2), y(3), -y(1)*y(3)/2];
```

Use these programs to generate a numerical solution, and use the numerical solution to answer the following questions:

1. Plot the velocity profiles \( u/U \) and \( v/\sqrt{U\nu/\bar{x}} \) versus \( \eta \).
2. Use the numerical solution to compute the dimensional vorticity as a function of \( x, y, U, \) and \( \nu \).
3. Obtain an expression for the vorticity at the plate and at the top of the boundary layer.
4. Plot the non-dimensional vorticity profile versus \( \eta \). Comment on the role of vorticity outside the boundary layer.
2.2.10 Suction Pile

Often, suction piles are used to anchor submarine risers. These are hollow tubes that are inserted into the sediments and then closed on the top so that they cannot be pulled out without removing the sediment trapped inside the pile. One way to set a submarine suction pile is by opening the top and letting the pile fall under its own weight and drive itself into the sediment. This problem explores the hydrodynamics of the pile during this free fall.

Although real piles are cylinders, we will simplify the problem as the two-dimensional flow between two flat plates (refer to Figure 1). The flow entering the pile is uniform across the gap 2b with velocity $U_0$. Boundary layers initially develop on the top and bottom plate and grow until the fully-developed flow region at C. Outside the boundary layers, between A and C, the velocity is uniform and equal to $U(x)$, but greater than $U_0$ so that the flow between the plates is steady.

1. (5 points) The flow per unit width entering the pile is $2bU_0$. Use this result and the solution for Couette flow presented in class (you do not need to derive it) to find an expression for the pressure gradient $dp/dx$ at C where the flow is fully developed (i.e., the boundary layers have merged). Write an expression for the velocity profile in this region in terms of $U_0$, b, and y.

2. (10 points) Approximate the velocity profile in the boundary layers between A and C as

$$u(y) = U \left( \frac{2y}{\delta} - \frac{y^2}{\delta^2} \right)$$  \hspace{1cm} (92)

Use this approximation to evaluate the ingredients to the von Kármán momentum integral $\delta^*, \theta$, and $\tau_0$.

3. (5 points) Recognizing that the flow at each cross section must be $2bU_0$ and using Equation 92, obtain an expression for $U(x)$ and $dU/dx$ between A and C.

4. (10 points) Solve the von Kármán momentum integral for $\delta(x)$. Note that the solution may be implicit.

5. (5 points) Use your solution to find the distance L between A and C. On what non-dimensional number does the quantity $L/b$ depend?
2.2.11 Blasius Solution

When estimating friction drag, a laboratory model of a ship haul can be approximated by a 20 cm wide by 60 cm long rectangular flat plate. Use Blasius solution to a laminar boundary layer to estimate the following for this flat plate in a steady, uniform outer current of both $U = 0.15$ and $U = 1.5$ m/s.

1. Plot the boundary layer thickness as a function of length along the plate for both outer flow conditions.

2. Plot the shear stress distribution along the length of the plate for both outer flow conditions.

3. Calculate the total friction drag force on the plate for both outer flow conditions.

4. Comment on similarities and differences between this laboratory model and the field prototype. Do you think the laboratory model will provide a means to estimate field-scale friction effects? If the model is used to test other phenomena, such as vessel stability in rough seas, do you think the friction effects will interfere with conclusions for these other flow dynamics? If so, for which types of motion and why? If not, why not?
2.3 Turbulence and Turbulent Boundary Layers
2.3.1 Turbulent Transport Equation

Apply Reynolds decomposition to the advective-reactive-diffusion equation

\[
\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ D \frac{\partial c}{\partial x_j} \right] - kc
\]

(93)

where \( k \) is a constant reaction rate constant. For the Reynolds decomposition, assume

\[
\begin{align*}
    u_i &= \bar{u}_i + u'_i \\
    c &= \bar{c} + c'
\end{align*}
\]

Substitute the Fick’s-law analogy for the turbulence closure model given by

\[
\overline{u'_j c'} = -D_t \frac{\partial \bar{c}}{\partial x_j}
\]

(94)

Finally, apply dimensional analysis in an open channel flow characterized by the depth \( h \) and shear velocity \( u_* \) to estimate a functional relationship for \( D_t \). Propose a method to measure the proportionality constant of your equation in the laboratory. Would these equations apply to concentration measured in the immediate vicinity of a chemical spill (distances less than 1 to 3 water depths).
2.3.2 Boundary Layer Drag Forces

A ship (120 m long and 30 m wide) leaves Port in Houston headed to the Panama Canal. Most of the haul is smooth except for a 2 m wide by 8 m long section on the center of the ship haul characterized by an equivalent sand roughness of 3 mm. Use the Blasius solution and 1/7-power relation to estimate the following:

1. At what speed does the ship haul drag transition from laminar to turbulent?

2. Estimate the total drag force at a velocity 5% and 25% of the transition velocity using relations from Blasius boundary layer.

3. Estimate the total drag force at a velocity twice and five times the transition velocity using relations from the 1/7-power relation.

4. Plot the total frictional drag force as a function of the Reynolds number for each of the estimates calculated above.

5. Comment on the apparent matching or mis-matching of these solutions near the transition velocity.

6. For the turbulent case, is the smooth-plate drag solution valid for the region in the wake of the roughened plate? Why or why not?
2.3.3 Ship bio-fouling

Comment on the validity of the following statement from a student recently captivated by the subject of scale analysis: “Barnacles and similar marine growth are unimportant on the surface of a ship since the size of these organisms is negligible relative to the size of the ship.” Base your comments on the following example.

The *M/V Viking Poseidon* (refer to Figure 12) is a multi-purpose offshore vessel based in Norway and mostly serving the North Sea. The table given below provides a few specifications. The Fuel Consumption data are for hydraulically smooth conditions.

Estimate each of the following:

1. Using Figure 3.12 from Newman (refer to course handouts), estimate the total drag on the ship when traveling at 12 knots.

2. For the same ship engine thrust, estimate the reduced speed of the *Viking Poseidon* when covered with barnacles of a typical scale $k = 2 \text{ mm}$. Assume that 10% of the engine thrust is used to overcome wave-making resistance.

3. Estimate the percent added fuel cost per 1000 NM when the ship is covered with barnacles compared to the hydraulically smooth vessel. Assume a diesel cost of $4.39 per gallon and that the engine thrust is that from part 2.

4. What is the range (maximum travel distance without refueling) of the ship under each condition (smooth and with barnacles)?
5. At what speed does the effect of the barnacles become negligible?

Based on these estimates, do you think it is worth the effort to keep the vessel free of barnacles?

*Viking Poseidon* specifications:

- Length : 93.35 m
- Breadth : 22.00 m
- Draught : 7.05 m
- Fuel capacity : 1355 m$^3$
- Fuel density : 0.870 g/cm$^3$
- Service Speed : 12 knots
- Fuel consumption : 17 tons/day at 12 knots
Figure 12: Photograph of the M/V Viking Poseidon owned by Eidesvik MPSV AS for Statoil serving the North Sea.

2.3.4 Ship Bio-fowling

Comment on the validity of the following statement from a student recently captivated by the subject of scale analysis: “Barnacles and similar marine growth are unimportant on the surface of a ship since the size of these organisms is negligible relative to the size of the ship.” Base your comments on the following example.

The M/V Viking Poseidon (refer to Figure 12) is a multi-purpose offshore vessel based in Norway and mostly serving the North Sea. The table given below provides a few specifications.

Viking Poseidon specifications:

Length : 93.35 m  
Breadth : 22.00 m  
Draught : 7.05 m  
Service Speed : 12 knots

Perform the following calculations twice, first assuming the hull is hydraulically smooth and second assuming the hull is covered with barnacles of a typical scale $k = 12$ mm.

1. At what speed does the ship hull drag transition from laminar to turbulent? Is it safe to assume the boundary layer is always turbulent?

2. Estimate the total friction drag force at a speed 1/3, 2/3 and 1 times the service speed. Use the 1/7-power relation and Figure 3.12 from Newman (1977) available in the lecture notes.

3. Plot the total frictional drag force as a function of the Reynolds number. Comment on the trend.
2.3.5 Short Answer

Answer the following questions clearly and succinctly. For problems requiring calculations, please show all your work. For problems requiring a deduction, state your reasoning.

1. List the fundamental assumption behind the Navier-Stokes equations for an incompressible fluid.

2. A 2 m long dolphin swims at 5 m/s. Where does the boundary layer on the dolphin’s skin transition from laminar to turbulent?

3. For flow of water around a 3 cm diameter cable, at what free-stream velocity is the Stokes approximation violated.

4. For the boundary layer on the hull of a ship, at what downstream distance from the front of the ship does the boundary layer approximation become valid?

5. What physics in a boundary layer is responsible for generation of vorticity?

6. Compute the local shear stress and total drag force on the bottom side of a flat plate 3 m long by 1 m wide at zero angle of attack immersed in a flow of water with outer velocity 1 cm/s. If you double the outer flow velocity, how much does the drag force increase.
2.3.6 Wind Boundary Layer

For a real wind over a reservoir, the boundary layer grows in the downstream direction as the wind propagates over the reservoir. Consider a wind for which the windspeed far outside the boundary layer is given by $U$. This developing boundary layer imposes a shear stress at the reservoir surface that causes the water to move. Assume the velocity deep in the reservoir remains at rest and that the reservoir is semi-infinite in the $x$-direction, originating at $x = 0$. For the boundary layers in the air and water, approximate the velocity profile by the parabolic function

$$\frac{u}{U} = a_0 + a_1 \frac{y}{\delta} + a_2 \left( \frac{y}{\delta} \right)^2$$

(95)

where $a_0$, $a_1$, and $a_2$ are constants and $u = 0$ at $y = 0$.

1. Write the boundary layer velocity profiles in the air and water with an appropriate form of Equation 95 so that the velocities match at the free surface. Note that the free surface boundary condition is also that $\tau$ is continuous at the free surface.

2. Compute the momentum thickness $\theta$, displacement thickness $\delta^*$, and the bottom shear stress $\tau_0$ in terms of $\delta$ from the given velocity profiles.

3. Solve for the boundary layer thickness in air and water using the von Kármán momentum integral and assuming the upstream boundary condition $\delta(x = 0) = 0$ for both air and water. Based on your solution, under what criteria is it acceptable to assume that the water surface velocity is zero when solving for the wind boundary layer?
2.3.7 Boundary Layer Separation

The von Kármán momentum integral can be used in combination with an outer flow solution around an object to identify possible points of boundary layer separation.

1. What criteria based on the velocity profile define the separation point?

2. Based on your answer above, is it possible for the parabolic profile given in Equation 95 to meet the separation criteria? Why or why not? If not, what alternative profile might you use?
3 Module 3: Ideal Flows

3.1 2D Potential Flows
3.1.1 Streamlines and Pressure

The complex potential for the uniform flow of an ideal fluid of velocity $U$ past a cylinder of radius $a$ that has a negative vortex of magnitude $\Gamma$ around it is given by

$$F(z) = U \left( z + \frac{a^2}{z} \right) + \frac{i\Gamma}{2\pi} \ln \frac{z}{a}$$  \hspace{1cm} (96)

where $z$ is the complex variable. To answer the questions below, use typical values of an ocean drilling shaft: $a = 0.25$ m, $U = 0.3$ m/s, and $\Gamma = 0.5$ m$^2$/s.

1. Extract the equation for the stream function $\psi$.

2. Plot the streamlines for values of $\psi$ equal to $\pm 0.01$, $\pm 1$, and $\pm 2$.

3. The velocity components on the surface of the cylinder are

$$\begin{align*}
u_R &= 0 \\
u_\theta &= -2U \sin \theta - \frac{\Gamma}{2\pi a} \end{align*}$$  \hspace{1cm} (97)

Start with the unsteady Bernoulli equation and the fact that $p = 0$ far from the cylinder and obtain an expression for the pressure on the surface of the cylinder.

4. Integrate the pressure field around the cylinder (numerically or analytically, whichever is most convenient) to show that there is a net lift force. What is the net force due to pressure acting on the top face of the cylinder between $\theta = 0$ and $\pi$?

5. What is the net lift force on a drill shaft in 500 m depth of water? Estimate the maximum friction and pressure drag from viscous flow theory assuming $\Gamma = 0$. What appears to be the dominant force to use in design of this shaft? What other forces are we neglecting?
3.1.2 Well Flow in an Underground Reservoir

The two-dimensional flow to a well in an underground reservoir is given by the solution for a sink between two confined walls, as sketched in Figure 13 (assuming only a small segment of the well is screened). To answer the questions below, use typical values for an oil well: $L = 30$ m, $Q = 0.1$ m$^2$/s,

1. Find the locations of the first two image sinks necessary to have the confining walls.

2. Each of the sinks defined above creates one wall, but also perturbs the opposite wall. Find the locations of four more image sinks necessary to cancel these perturbations.

3. Plot the streamlines for this flow field with all 6 image sinks included in the solution. Is this an adequate number of images? Why or why not?

4. If the pressure is zero far from the well, find an expression for the pressure along a streamline down the center of the reservoir. What would you expect the pressure to be at the entrance to a well if the well diameter is 0.2 m? Why does the pressure diverge at the center of the well for this potential flow solution?

5. What is the maximum velocity of the approach flow at 1 m distant from the well centerline?
3.1.3  Propeller Wash

The propeller wash from a ship is sometimes represented by the superposition of a counterclockwise rotating vortex and a source.

1. Write down the expressions for the complex potential $F$ and complex velocity $W$ for the vortex magnitude $\Gamma$ and source magnitude $m$.

2. Using polar coordinates, derive an expression for the velocity $u_R$ and $u_\theta$ in the propellor wash versus distance $r$ from the center. What is the ratio of $u_R$ to $u_\theta$?

3. If the propellor is located a distance $L$ above the bottom of a channel, use appropriate images to predict the maximum velocity at the channel bed.

4. What information would you need to gather to use these equations to determine whether the propeller wash from a ship is likely to scour the bed in a certain channel?
3.1.4 Superposition

Show that a doublet can also be obtained by the superposition of a clockwise vortex at $y = \epsilon$ and a counterclockwise vortex at $y = -\epsilon$ by letting $\epsilon \to 0$. 
3.1.5 Hurricane Winds

Consider a Category 5 hurricane that has a maximum wind speed of 250 kph at the eye wall, located 15 km from the center of the hurricane. If the flow in the hurricane outside the hurricane’s eye is approximated as a free vortex, do the following:

1. Determine the wind speeds at locations 25, 50, and 75 km from the center of the storm.

2. Determine the center pressure at the eye wall (at \( r = 15 \) km). Assume that far from the hurricane, the wind is zero and the pressure is \( p_\infty = 101 \) kPa. Also assume that the air is of constant density at \( \rho_a = 1.2 \) kg/m\(^3\).
3.1.6 Sink Capture Zone

A model for the coolant water intake for a boat engine is approximated by a uniform flow $U$ of an ideal fluid passing from left to right over a plate at $y = 0$ with a sink of magnitude $Q$ located at $x = 0$.

1. Write down the complex potential and complex velocity for this flow field. Do not forget the image source necessary to create the solid wall at $y = 0$.

2. Downstream of the well, there is a stagnation point on the wall at $y = 0$. Locate the $x$-coordinate of the stagnation point.

3. Evaluate the streamfunction at the stagnation point and obtain an expression for the stagnation streamline. What is the maximum height $y = H$ that this streamline reaches above the $x$-axis?

4. Integrate $U$ from $y = 0$ to $H$ to show that all of the fluid below this streamline eventually flows into the well.
3.1.7 Blasius Integral Laws

Use Blasius integral laws to compute the force and moment on the following objects in 2D potential flows. The $x$- and $y$-axes are the horizontal and vertical axes, respectively, and the $z$ axis is out of the page.

1. A cylinder (axis along the $z$-axis) having clockwise rotation in a uniform flow in the negative $y$-direction.

2. A 2D Rankine ovoid given by a source at $y = a$, a sink of equal magnitude at $y = -a$ and a uniform current in the negative $y$-direction.

Does the free stream velocity affect the net force in the cases above? How would your results be different in a real flow (one that includes friction)?
3.1.8 Sand Grains

A solid hemisphere of radius $a$ is lying on a flat plate. A uniform flow of velocity $U$ is streaming past it. Do the following

1. Find an expression for the velocity potential $\phi$.

2. Use Bernoulli’s equation to find an expression for the pressure on the outer surface of the hemisphere. Assume gravity acts downward and normal to the flat plate.

3. What is the pressure on the hemisphere on the bottom surface touching the plate?

4. Integrate the pressure over the surface of the sphere to show that the density $\rho_s$ of the sand grain must be

$$\rho_s \geq \rho \left( 1 + \frac{33 U^2}{64 ag} \right)$$

for the sand grain to stay on the plate.

5. For silica sand the density is about $\rho_s = 2.4$ g/cm$^3$. For a course sand of diameter $2a = 2.5$ mm, what velocity of flow will just lift the particle? Does this velocity agree with your intuition? What other factors are active in sediments for real environmental applications?
3.1.9 Image Sources

Solid boundaries in potential flow can be created by superposition using the method of images. In this method, sources and sinks of equal magnitude are placed equal distances from the desired boundary to generate the boundary. To illustrate this point, consider a source located a height $h$ above a solid wall.

1. Write the complex potential for a 2D-source of magnitude $Q$ located at $y = h$ in the complex plane superposed with an equal-magnitude source at $y = -h$.

2. Extract the velocity potential and stream function from the complex potential.

3. Show that the line $y = 0$ (coincident with the $x$-axis) is a streamline.

4. Extract the components of the velocity. What is the value of the vertical component of the velocity at $y = 0$? Does the flow in the half-plane $y \geq 0$ represent flow from the source a distance $h$ above a solid boundary at $y = 0$?

5. Explain how you would use image sources to create a sink with clockwise circulation a distance $l$ to the right of a vertical wall at $x = l/2$. 
3.1.10 Conformal Mapping

Consider the flow of a uniform stream of speed $U$ at an angle of attack $\alpha$ past an ellipse of semi-axes $a$ and $b$.

1. Show that the transformation

$$z = \zeta + \frac{a^2 - b^2}{4\zeta}$$

maps the ellipse into a circle of radius $(a + b)/2$ in the $\zeta$-plane.

2. Sketch the streamlines of the flow (by hand is fine). What are the values of $\gamma$ for the stagnation points in the $\zeta$-plane. Use these values plus the transform in Equation (99) to find the stagnation points on the ellipse.

3. Plot (using Matlab or a similar numerical package) the pressure distribution on the ellipse (i.e., plot $P(\theta)$) for $a = 1$ m, $b = 0.25$ m, $U = 1$ m/s, and $\alpha = 30^\circ$. 


3.1.11 A vortex in a bounded free stream

The complex potential for a clockwise vortex at $y = h$ above an infinite plane at $y = 0$ in a uniform flow in the $x$-direction of magnitude $U$ is given by

$$F(z) = Uz + \frac{i\Gamma}{2\pi} \left[ \ln(z - ih) - \ln(z + ih) \right]. \quad (100)$$

1. Compute the complex velocity.
2. Obtain the velocity potential and the stream function.
3. Obtain the velocity components $(u, v)$.
4. Find the locations of any stagnation points on the $x$-axis.
5. If stagnation points exist on the $x$-axis, what might that imply about what this flow field represents? i.e., what is the physical implication of their existence?
3.2 3D Potential Flows
3.2.1 Sphere in the Flow Field of a Doublet

Consider a three-dimensional, axi-symmetric potential flow having a doublet of strength $\mu$ located at $x = l$ and a doublet of strength $-a^3\mu/l^3$ located at $x = a^2/l$. Show that $\psi = 0$ on the surface $r = a$; hence, demonstrate that this creates a sphere of radius $a$ centered at $x = 0$ in the flow field of a doublet. If there is a net force on the sphere caused by the doublet at $x = l$, does this violate D’Alembert’s paradox? Why or why not?
3.2.2 Tidal Flows at Inlets

When the ebb tide flows through a narrow inlet, as on a barrier island, a vortex dipole may form on the open-coast side of the inlet. This vortex pair can be modeled as the two-dimensional potential flow of two counter-rotating vortices:

1. Write down the complex potential for a counter-clockwise rotating vortex of magnitude $\Gamma$ located at $y = -L/2$ and a clockwise rotating vortex of equal magnitude located at $y = L/2$. You may neglect the presence of the barrier islands, that is, do not include any image vortices.

2. Extract an equation for the velocity potential and stream function.

3. Write down the complex velocity for this flow field.

4. Extract an equation for the velocity components $u$ and $v$.

5. Use an appropriate Bernoulli’s equation to evaluate the pressure at $y = 0$ and $x = \pm L$. Let the pressure equal zero at $x = \pm \infty$, where the flow is also stagnant.

6. Use the Blasius integral laws to obtain an expression for the force on the vortex pair by considering a control surface that encloses both vortices.

7. When the tide reverses, the vortex pair can be approximated to experience a uniform flow of velocity $U$ in the positive $x$-direction. Add this component to the model and evaluate the location(s) of any stagnation points along the $x$-axis. Again, neglect the barrier islands.
3.2.3 Added-Mass Coefficients

For each of the objects in Figure 14, determine which of the added mass coefficients will be non-zero and which will be zero. Explain your reasoning. Note that you do not need to compute the added mass coefficients.
3.2.4 Rankine Objects

Determine the stream function for a simple axisymmetric Rankine body that is 3 m long and 2 m maximum in diameter in a uniform stream of water at 3 m/s. Find the pressure at the stagnation points and their locations.
3.2.5 Basic Solutions to Potential Flow Problems

Write down the complex potential for the following two-dimensional planar ideal flow problems. For both problems, please express the complex potential in terms of the complex variable $z$.

1. A sink flow of magnitude $Q$ in a uniform flow in the $x$-direction of magnitude $U$ against an infinite plane at $y = 0$. This is the model of a cooling water intake on the bottom of a ship—make sure that the full flow rate $Q$ comes from the half-plane below the ship.

2. A cylinder of radius $a$ in a uniform flow of magnitude $U$ flowing at an angle $\alpha$ to the $x$-axis.

Write down the velocity potential for the following three-dimensional axisymmetric ideal flow problems.

1. A doublet at $x = -L$ in a uniform flow in the $x$-direction of magnitude $U$ with a source of magnitude $Q$ at $x = 0$ and a sink of magnitude $Q$ at $x = L$. Please use Cartesian coordinates.

2. A sphere of radius $a$ in a uniform flow in the $(-x)$-direction of magnitude $U$. Please use spherical coordinates.
3.2.6 Moving Sphere

In this problem we consider the forces on a sphere under different types of motion. Let gravity be in the \((-y)\)-direction. For all parts of the problem, consider the pressure at \(x = \infty\) and \(y = z = 0\) to be zero.

1. The velocity potential for a stationary sphere of radius \(a\) in a uniform flow to the right of magnitude \(U\) is given by

\[
\phi = U \cos \theta \left( r + \frac{a^3}{2r^2} \right)
\]

Find the locations and pressure of the stagnation points of the flow. Based on this calculation, do you expect a net force on the cylinder in the \(x\)-direction? Why or why not?

2. Obtain the velocity potential for a sphere of radius \(a\) moving at a velocity \(U\) along the \(x\)-axis in a quiescent fluid. Hint: use the expression above, look at the velocity components and determine how to remove the velocity at infinity while preserving the velocity near the sphere.

3. For the moving sphere, obtain the pressure at \(x = \pm a\) on the surface of the sphere. How does it compare to the pressure obtained above? Do you expect a change in the net force on the sphere in the \(x\)-direction? Why or why not?

4. Finally, consider an accelerating sphere with velocity \(U(t)\) and acceleration \(\dot{U}(t)\). Write down the velocity potential. Obtain the pressure at \(x = \pm a\) on the surface of the sphere under this condition. Do you expect a net hydrodynamic force in the \(x\)-direction for this case based on this calculation? If this is different from the cases of constant velocity studied above, where is the difference coming from, i.e., what physical process is responsible for the difference?
3.2.7 Rankine Half-Body

Consider a uniform flow in the $x$-direction superposed with a three-dimensional source at the origin.

1. Find an equation for the dividing stream tube that outlines the Rankine half-body created by the source in this flow field.

2. Find the location of the stagnation point.

3. Find the radius of the body in the plane $x = 0$.

4. Find the maximum radius of the body far downstream.

5. Find an expression for the tangential velocity on the surface of the body, and plot this velocity as a function of $x$. 
Module 4: Buoyancy-Driven Flows