Two-Dimensional Potential Flow and the Stream Function

Learning Objectives:
1. Write and explain the fundamental equations of potential flow theory
2. List and explain the assumptions behind the classical equations of fluid dynamics

Topics/Outline:
1. Introduce the velocity potential and the stream function
2. Derive the governing equations for 2D and axi-symmetric potential flow
3. Solve 2D potential flow equations
4. Introduce some complex algebra
5. Example of a uniform 2D current

Reading:
Sections 4.1-4.3
Section 6.1-6.4
Velocity Potential:
Define a scalar velocity potential function $\phi$ such that
\[ \vec{u} = \nabla \phi \]
$\phi$ defines $\vec{u}$; possible in any fluid.

For an ideal fluid:
1) Conservation of Momentum (Kelvin's theorem):
\[ \nabla \times \vec{u} = 0 \text{ everywhere} \]
\[ \nabla \times \nabla \phi = 0 \]
\[ \nabla \times \nabla \phi = 0 \quad \text{true for any } \phi \text{ by vector identity.} \]
\[ \Rightarrow \phi \text{ satisfies automatically.} \]

2) Conservation of Mass:
\[ \nabla \cdot \vec{u} = 0 \]
\[ \nabla \cdot \nabla \phi = 0 \]
\[ \nabla^2 \phi = 0 \quad \text{: homogeneous Laplace equation.} \]

Thus, the governing equation for $\phi$ for an ideal fluid is the Laplace Equation.
A flow described by $\phi$ such that $\vec{u} = \nabla \phi$ is called a potential flow.
Stream Function:

Define a vector stream function $\vec{\psi}$ such that

$$\vec{u} = \nabla \times \vec{\psi}$$

In general, this is quite complicated unless $\vec{\psi}$ has only one component. This occurs in Axi-symmetric 3D flow and in all 2D flows. Then,

$$\vec{\psi} = (0, 0, \psi)$$

$\psi$ component normal to components of the velocity vector.

1) Conservation of Mass:

$$\nabla \cdot \vec{u} = 0$$

$$\nabla \cdot (\nabla \times \vec{\psi}) = 0$$

true for any $\vec{\psi}$ by vector identity

$\vec{\psi}$ satisfies automatically.

2) Conservation of Momentum:

$$\nabla \times (E_{ijk} \frac{\partial \psi_k}{\partial x_j}) = 0$$

$$E_{ijk} \frac{\partial}{\partial x_i} (E_{ijk} \frac{\partial \psi_k}{\partial x_j}) = 0$$

$$(\delta_{ij} \delta_{pk} - \delta_{ik} \delta_{pj}) \frac{\partial^2 \psi_k}{\partial x_i \partial x_j} = 0$$

$$\delta_{ij} \delta_{pk} \frac{\partial^2 \psi_k}{\partial x_p \partial x_i} - \delta_{ik} \delta_{pj} \frac{\partial^2 \psi_k}{\partial x_p \partial x_j} = 0$$

$$\frac{\partial^2 \psi_k}{\partial x_j \partial x_i} - \frac{\partial^2 \psi_k}{\partial x_i \partial x_j} = 0$$
only $\Psi_3$ has a value:

$$
\frac{\partial}{\partial x_3} \left( \frac{\partial \Psi_3}{\partial x_1} + \frac{\partial \Psi_3}{\partial x_2} + \frac{\partial \Psi_3}{\partial x_3} \right) - \left( \frac{\partial^2 \Psi_3}{\partial x_1^2} + \frac{\partial^2 \Psi_3}{\partial x_2^2} + \frac{\partial^2 \Psi_3}{\partial x_3^2} \right) = 0
$$

$q=1$:

$$\frac{\partial}{\partial x_1} \left( \frac{\partial \Psi_3}{\partial x_3} \right) - \phi = 0$$

$\phi$: for 2D or axi-sym. flow $\Psi_3 = f(x_1, x_2)$

$\checkmark$ automatic

$q=2$:

$$\frac{\partial}{\partial x_2} \left( \frac{\partial \Psi_3}{\partial x_3} \right) - \phi = 0$$

see above

$$\frac{\partial}{\partial x_3} \left( \frac{\partial \Psi_3}{\partial x_3} \right) - \frac{\partial^2 \Psi_3}{\partial x_1^2} = 0$$

Must be true to match momentum conservation (Kelvin's Theorem)

Thus, $\Psi_3 = \Psi$ must satisfy

$$\nabla \nabla \Psi = 0$$

$$\nabla^2 \Psi = 0$$

homogeneous Laplace Eqn.
Two-dimensional Potential Flow:

\( \vec{u} = (u, v) \) : then the \( \phi \) and \( \Psi \) definitions become

\[
\begin{align*}
  u &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\
  v &= -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}
\end{align*}
\]

\( \phi \) and \( \psi \) satisfy Laplace Equation.

\( \uparrow \) functions meeting these criteria are solutions to the Cauchy-Riemann Equations.

Solutions to CR equations are any 2D complex analytical function:

\[ F(z) = \phi(x, y) + \psi(x, y)i \]

complex variable function \( z = x + iy \)

\[ F(z) \] is the complex potential.

Obtain the velocity from \( \frac{dF}{dz} \):

\[
W(z) = \frac{dF}{dz} = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial x} i
\]

\[ = u - vi \]

\( W(z) \) is the complex velocity.
Useful Results:

$$W W^* = (u - iv)(u + iv)$$

complex conjugate

$$= u^2 + ivu - ivu - i^2v^2$$

$$= u^2 + v^2 = \nabla \phi \cdot \nabla \phi$$

$$\frac{dA}{dz} = \frac{dA}{dx}$$

cylindrical coordinates:

$$\hat{u} = (u, v)$$

$$u_r = \frac{\partial \phi}{\partial r}$$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

transform:

$$u = u_r \cos \theta - u_\theta \sin \theta$$

$$v = u_r \sin \theta + u_\theta \cos \theta$$

Complex Variables:

$$z = re^{i\theta} = r \cos \theta + ir \sin \theta$$

$$F(z) = \phi + i \psi$$

$$W(z) = u - iv = (u_r - iu_\theta)e^{-i\theta}$$
Example: 2D Uniform Flow.

2D potential flow solutions: Guess a function $F(z)$, extract the flow field, decide what the flow field really is. Hence, we get the flow field after we choose a function.

Choose $F \propto z$

For the most general flow, choose a complex proportionality coefficient:

$$F = ce^{-ia}z$$

Write out $F(z)$:

$$F(z) = c(e^{ia}(x + iy))$$

$$= c[x cos \alpha - i^2 y sin \alpha + i(y cos \alpha - x sin \alpha)]$$

$$= c(x cos \alpha + y sin \alpha) + ic(y cos \alpha - x sin \alpha)$$

Get complex velocity:

$$W(z) = \frac{dF}{dx} = c \cos \alpha - ic \sin \alpha$$

Find velocity:

$$U = c \cos \alpha; \quad V = c \sin \alpha$$

$c =$ magnitude of velocity.

$$F = |U|e^{-ia}z$$

$c$ uniform flow of velocity $U$ at angle $\alpha$ to the $x$-axis.
<table>
<thead>
<tr>
<th>Velocity Potential</th>
<th>Stream Function</th>
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</thead>
<tbody>
<tr>
<td><strong>Definition:</strong> (No assumption)</td>
<td>$\nabla \phi = \bar{u}$</td>
</tr>
<tr>
<td>Conservation of Mass: (incompressible flow)</td>
<td>$\nabla \cdot \bar{u} = 0$</td>
</tr>
<tr>
<td>$\nabla^2 \phi = 0$</td>
<td>$\nabla \cdot (\nabla \times \bar{\psi}) = 0$ (Satisfied automatically)</td>
</tr>
<tr>
<td><strong>3D Laplace Equation</strong></td>
<td>(Inertial flow)</td>
</tr>
<tr>
<td>Conservation of Momentum:</td>
<td>$\nabla \times \bar{u} = 0$</td>
</tr>
<tr>
<td>$\nabla \times \nabla \phi = 0$</td>
<td>$\nabla \times \nabla \times \bar{\psi} = 0$</td>
</tr>
<tr>
<td><strong>Satisfied automatically</strong></td>
<td>Generally rather complicated.</td>
</tr>
</tbody>
</table>

2D Potential Flow:

$\phi$ and $\psi$ satisfy the Laplace equations.

$u$: \[
\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}
\]

$v$: \[
\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}
\]

Called Cauchy-Riemann equations. (any)

Solution $\rightarrow$ Complex analytic func.