Three-Dimensional Potential Flows

Learning Objectives:

1. Write and explain the fundamental equations of potential flow theory
2. Compute the flow field around 2D and 3D objects using combinations of fundamental potential flow solutions

Topics/Outline:

1. Coordinate System
2. Governing equations for the velocity potential
3. Solution for the velocity potential
4. Stokes Stream function

Reading:

Section 5.1-5.3

Section 6.17-6.18

Section 4.6-4.6
Axi-Symmetric Potential Flow:

Coordinate System:

Axisymmetric:
\[ \frac{\partial}{\partial r} \phi = 0 \]

\[ (r, \theta, z) \rightarrow (r, \theta) \]

\[ \vec{u} = (0, 0, \omega_z) \]

\[ \Psi = (0, 0, \Psi) \]

Velocity Potential:

\[ \vec{u} = \nabla \phi \]

in spherical coordinates:

\[ u_r = \frac{\partial \phi}{\partial r} \]

\[ u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \]

Conservation of Momentum:

\[ \nabla \times \vec{u} = 0 \rightarrow \nabla \times \nabla \phi = 0 \]

satisfied by vector identity.
Conservation of Mass:
\[ \nabla \cdot \mathbf{u} = 0 \rightarrow \nabla^2 \phi = 0 \] : Laplace Equation.

in spherical coordinates:
\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) = 0 \]

substitute velocity potential:
\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\partial \phi}{\partial \theta} \sin \theta \right) = 0 \]

Solution for \( \phi \): Separation of variables:

Let:
\[ \phi = R(r) T(\theta) \]

Substitute:
\[ \frac{\partial \phi}{\partial r} = \frac{\partial}{\partial r} (R(r) T(\theta)) = R \frac{\partial T}{\partial r} + T \frac{\partial R}{\partial r} \]
\[ = T \frac{\partial R}{\partial r} \]

\[ \frac{\partial \phi}{\partial \theta} = \frac{\partial}{\partial \theta} (R T) = R \frac{\partial T}{\partial \theta} + T \frac{\partial R}{\partial \theta} \]
\[ = R \frac{\partial T}{\partial \theta} \]

Then:
\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T \frac{\partial R}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( R \frac{\partial T}{\partial \theta} \sin \theta \right) = 0 \]
\[
\frac{r^2}{TR} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R}{TR} \frac{r^2}{x^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) = 0
\]

pure function of \( r \): \( 0 \rightarrow a \)

pure function of \( \theta \): \( 0 \rightarrow d \)

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = -\frac{1}{T \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right)
\]

pure function of \( r \)

pure function of \( \theta \)

To have equality for any arbitrary point \( x = (r, \theta) \), both sides must equal the same constant \( C \).

Classically, we let \( C = \lambda (l+1) \)

\( \lambda \) is constant.

1.) Solution for \( R(r) \):

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \lambda (l+1)
\]

Solution is \( R = Kr^\alpha \) with \( \alpha = \lambda \) or \( \alpha = -(l+1) \)

General solution is linear combination:

\[
R(r) = A_\lambda r^\lambda + \frac{B_\lambda}{r^{l+1}}
\]
2.) Solution for $T(\theta)$:
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dT}{d\theta} \right) + \ell (\ell + 1) T = 0
\]

Substitute (change variable to) $x = \cos \theta$:
\[
\frac{d}{dx} \left( (1 - x^2) \frac{dT}{dx} \right) + \ell (\ell + 1) T = 0
\]

This is Legendre's Equation. Solution is:
\[
T_l(x) = C_l P_l(x) + D_l Q_l(x)
\]

function of $x$

$\Rightarrow$ convert to $\theta$

\[
T_l(\theta) = C_l P_l(\cos \theta) + D_l Q_l(\cos \theta)
\]

function argument

$P_l$: Legendre's function of the 1st kind. $P_l(\pm 1)$ diverges for $\ell \neq$ integer
$\Rightarrow \ell$ must be an integer.

$Q_l$: Legendre's function of the 2nd kind. $Q_l(\pm 1)$ diverges for all $\ell$
$\Rightarrow D_l = 0$.

\[
T_l(\theta) = C_l P_l(\cos \theta).
\]
Collecting the solution:

\[ \phi = (A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}) P_\ell (\cos \theta) \]

\[
P_\ell (x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell
\]

\[
P_0 (x) = 1
\]

\[
P_1 (x) = x
\]

\[
P_2 (x) = \frac{1}{2} (3x^2 - 1)
\]
Stream Function:

\[ \mathbf{\bar{u}} = \nabla \times \nabla \psi \]

For axi-symmetric flow, define Stoke's Stream Function:

\[ u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} \]

\[ u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \]

Conservation of Mass: Spherical coordinates:

\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) = 0 \]

\( L \) satisfies automatically.

Conservation of Momentum: does not lead to Laplace equation for \( \psi \).

\( \psi \): is not the solution to a Laplace equation.

\( L \) solutions are fundamentally different from 2D flows.

Find \( \psi \) from \( \phi \) through \( \mathbf{\bar{u}} \).

Physical interpretation: \( \psi \) is stream surface.

\[ Q = 2\pi (\psi_2 - \psi_1) \]

\( \uparrow \) flux through region bounded by to stream tubes.