Blasius Boundary Layer Solution

Learning Objectives:
1. Develop approximations to the exact solution by eliminating negligible contributions to the solution using scale analysis

Topics/Outline:
1. Identification of similarity solution for Blasius boundary layer
2. Substitution of similarity solution into boundary layer equations
3. Simplification to a single ODE
4. Numerical solution to Blasius boundary layer equation

Reading:
  Chapter 9
  Chapter 10
  Chapter 3
  Chapter 12
Similarity Solution to Blasius Boundary Layer:

Since $x$ and $y$ are semi-infinite, there is no characteristic geometric scale.
Thus, we may expect a similarity solution. Indeed, the velocity profile maintains a constant shape:

In non-dimensional space, our equations were:

$$x^* = \frac{x}{L} \quad \quad u^* = \frac{u}{V}$$

$$y^* = \frac{y}{\delta} \quad \quad v^* = \frac{v}{V} = \frac{V}{V^2/L}$$

Giving:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{1}{\delta^2} \frac{\partial^2 u^*}{\partial y^*^2}$$

Both terms must be of order 1.
Thus, we deduce:

\[ \frac{1}{Re_L} \frac{L^2}{\delta^2} \approx 1 \]

↓ solve for \( \delta \):

\[ \delta = \sqrt{\frac{2L}{U}} \]

Substitute \( \delta \) and \( L = x \) into our scale definitions:

\[ x^* = \frac{x}{x} = 1 \]

\[ u^* = \frac{u}{U} \]

\[ y^* = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x/U}} \]

\[ v^* = \frac{v}{\sqrt{\nu y^*}} \]

We have eliminated the dependence on \( x^* \).

\( y^* \) is our similarity variable.

Our similarity solution will give:

\[ \eta = \frac{y}{\sqrt{\nu x/U}} \]

And all data at any \( x \) and \( y \) will collapse to these curves.
Blasius Boundary Layer:

We assume a solution of the form:

\[
\begin{align*}
  u &= U \, F(\eta) \\
  v &= \sqrt{\frac{V}{x}} \, G(\eta) \\
\end{align*}
\]

with \( \eta = \frac{y}{\sqrt{\frac{V}{x}}} \).

For the chain rule, we will need:

\[
\begin{align*}
  \frac{\partial y}{\partial x} &= \frac{2}{\sqrt{V}} \left( y \sqrt{\frac{V}{x}} \right) = -\frac{1}{2} y \sqrt{\frac{V}{x}} \frac{1}{x} = -\frac{\eta}{2x} \\
  \frac{\partial y}{\partial \eta} &= \frac{2}{y} \left( y \sqrt{\frac{V}{x}} \right) = \sqrt{\frac{V}{x}}
\end{align*}
\]

We also need:

\[
\begin{align*}
  \frac{\partial u}{\partial x} &= \frac{2}{\sqrt{V}} \left( U F(\eta) \right) = U \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial x} = -\frac{U \eta}{2x} \frac{\partial F}{\partial \eta} \\
  \frac{\partial u}{\partial y} &= \frac{2}{\sqrt{V}} \left( U F(\eta) \right) = U \frac{\partial F}{\partial \eta} \frac{\partial \eta}{\partial y} = U \sqrt{\frac{V}{x}} \frac{\partial \eta}{\partial y} \\
  \frac{\partial v}{\partial y} &= \frac{2}{\sqrt{V}} \left( \sqrt{\frac{V}{x}} \, G(\eta) \right) = \sqrt{\frac{V}{x}} \frac{\partial G}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{V}{x} \frac{\partial G}{\partial \eta}
\end{align*}
\]

Boundary Conditions:

@ \( y = 0 \):
\[
\begin{align*}
  u(x,0) &= 0 \quad \rightarrow \quad F(0) = 0 \\
  v(x,0) &= 0 \quad \rightarrow \quad G(0) = 0
\end{align*}
\]

Match:
\[
\begin{align*}
  u(x,\infty) &= u \quad \rightarrow \quad F(\infty) = 1
\end{align*}
\]
Conservation of Mass:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

\[ \frac{v}{x} \left[ -\frac{\eta}{2} \frac{\partial F}{\partial \eta} + \frac{\partial G}{\partial \eta} \right] = 0 \]

\[ \frac{dG}{d\eta} = \frac{\eta}{2} \frac{df}{d\eta} \]

ODE since \( G, F \) are functions of \( \eta \) only.

To solve for \( G \), introduce a new unknown function \( f \):

\[ F = \frac{df}{d\eta} \quad ; \quad f = \text{slope of } F \]

\[ = \text{definition of a streamline.} \]

\[ \implies f \text{ is the stream function.} \]

Then, substituting and integrating:

\[ \int \frac{dG}{d\eta} d\eta = \int \frac{\eta}{2} \frac{df}{d\eta^2} d\eta \]

Integrate by parts:

\[ \int uv = uv - \int v du \]

\[ u = \frac{\eta}{2} \quad \text{dv} = \frac{df}{d\eta^2} \]

\[ du = \frac{d\eta}{2} \quad v = \frac{df}{d\eta} \]

\[ G = \frac{\eta}{2} \frac{df}{d\eta} - \int \frac{1}{2} \frac{df}{d\eta} d\eta \quad + C \]

Choose \( C \); only affects the magnitude of the stream function.

\[ G = \frac{1}{2} \left( \eta \frac{df}{d\eta} - f \right) \]
Conservation of Momentum:

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) + \sqrt{\frac{\partial v}{\partial x}} G(\eta) \sqrt{\frac{\partial v}{\partial x}} \frac{\partial F}{\partial \eta} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \eta} \right) \]

Substitute:

\[ F = \frac{dF}{d\eta} \quad \text{and} \quad G = \frac{1}{2} \left( \eta \frac{dF}{d\eta} - f \right) \]

Giving:

\[ \frac{1}{2} \left( \eta \frac{dF}{d\eta} - f \right) \frac{d^2 F}{d\eta^2} - \frac{\eta dF}{d\eta} \frac{d^2 F}{d\eta^2} = \frac{d^3 F}{d\eta^3} \]

\[ \frac{d^2 F}{d\eta^2} + \frac{f d^2 F}{2 d\eta^2} = 0 \]

Boundary Conditions:

\[ F(0) = 0 \quad \rightarrow \quad \frac{dF}{d\eta} = 0 \quad \text{at} \quad \eta = 0 \]

\[ G(0) = 0 \quad \rightarrow \quad f = 0 \quad \text{at} \quad \eta = 0 \]

\[ F(\infty) = 1 \quad \rightarrow \quad \frac{dF}{d\eta} = 1 \quad \text{at} \quad \eta = \infty \]
Solution:

Solve for $f$ using 4th order Runge-Kutta scheme.

Convert to system of ODE's:

$$
\frac{f}{2} \frac{d^2 f}{d\eta^2} + \frac{d^3 f}{d\eta^3} = 0
$$

Define:

$$
F = \frac{df}{d\eta}
$$

$$
H = \frac{dF}{d\eta} = \frac{d^2 f}{d\eta^2}
$$

Then we have:

$$
\frac{dH}{d\eta} = -\frac{f}{2} H
$$

$$
\frac{dF}{d\eta} = H
$$

$$
\frac{df}{d\eta} = F
$$

Subject to the conditions:

$$
f(0) = 0
$$

$$
F(0) = 0
$$

$$
F(\infty) = 1
$$

Solution is considered "Exact" since equation is an ODE and numerical method has very low error.