Law of the Wall and the Velocity Defect Law

Learning Objectives:

1. Formulate models for turbulent flow problems using Reynolds decomposition

Topics/Outline:

1. Viscous Sub-layer
2. Inertial Sub-layer
3. Log-Law of the Wall
4. Outer Layer

Reading:

Chapter 4
Chapter 3
Chapter 5
Viscous Sub-Layer:

Thin layer so close to a thin wall that turbulence cannot penetrate into it:

\[ \overline{uv} \rightarrow 0. \]  (However, there are still instantaneous pressure and velocity fluctuations)

Solution:

\[ \nu \frac{\partial \overline{u}}{\partial y} = \frac{\partial \overline{u}}{\partial y} \]

\[ \frac{\partial \overline{u}}{\partial y} = \frac{u_x^2}{\nu} dy \]

\[ u_y^{+} = \left. \frac{u_x^2 y}{\nu} \right|^{y}_{0} \]

\[ \frac{u(y)}{u^{+}} = y^{+} \]  (Non-dimensional velocity profile is linear.)

For a rough bed:

No-slip does apply here.

B.C. for \( \overline{u} \) on \( y^+ = 0 \) is NOT a no-slip condition.

If rms roughness height is \( \Theta(y^+ = 5) \), there is no viscous sublayer

Use rough solutions.
Inertial Sub-Layer:

The inner and outer solutions must match in this region:

\[ \frac{\tilde{u}}{u_*} = f(y^+) \quad , \quad \eta \ll 1 \]

\[ \frac{\Delta \tilde{u}}{u_*} - \frac{v - \tilde{u}}{u_*} = g(\eta) \quad , \quad y^+ \gg 1 \]

Solve each for \( \tilde{u} \):

\[ \tilde{u} = u_* f \left( \frac{u_* y}{v} \right) \]

\[ \tilde{u} = -u_* g \left( \frac{y}{\delta} \right) + v \]

Apply \( y^+ f(y^+) \) to each equation:

\[ y^+ \frac{\partial \tilde{u}}{\partial y} = \frac{u_*^2}{v} y f'(y^+) = -\frac{u_*}{\delta} y g'(\eta) \]

\[ \sqrt{\frac{\nu}{\nu}} y^+ f'(y^+) = -u_* \eta g'(\eta) \]

Since \( y^+ = \frac{u_* y}{v} \) and \( \eta = \frac{y}{\delta} \) are independent (you may vary one while the other stays constant):

\[ y^+ f'(y^+) = -\eta g'(\eta) = \text{constant} = \frac{1}{K} \]

"kappa": known as the von Karman constant.

Then:

\[ f'(y^+) = \frac{1}{Ky^+} \]

\[ g'(\eta) = \frac{1}{K\eta} \]
Integrating, we have:

\[
\frac{\bar{u}}{u_*} = \frac{1}{K} \ln(y^+) + A
\]

\[
\frac{U - \bar{u}}{u_*} = -\frac{1}{K} \ln(\eta) + B
\]

↑ Logarithmic velocity profile
Called the Log-Law of the Wall.
Only strictly valid in the inertial sub-layer.

We may manipulate these equations to eliminate \( \bar{u} \):
Add them together:

\[
\frac{U}{u_*} = \frac{1}{K} \ln\left(\frac{y^+}{\eta}\right) + A + B
\]

\[
\frac{U}{u_*} = \frac{1}{K} \ln\left(\frac{u_* \delta}{\nu}\right) + A + B
\]

↑ relate \( u_* \) to \( U \) and \( \delta \).
Outer Layer:

The solution in the outer layer comes from
\[ \frac{U-u}{u_*} = g(\eta) \]

And must converge on inertial sub-layer solution as \( \eta \ll 1 \).

Traditionally, we write:

\[ \frac{U-u}{u_*} = -\frac{1}{k} \ln(\eta) + B - W(\eta) \]

\( W(\eta) \): called the wake function and represents the difference between the defect law and the log law.

Experimental Results:

\( K, A, B, (W(\eta)) \) can only be found by experiment.

\[ K \approx 0.38 \text{ to } 0.43 \rightarrow 0.4 \]
\[ A \approx 5.5 \]
\[ B \approx 1.0 \]

Constants work well in a wide range of wall-bounded shear flows (pipes, channels, flat-plate). (See Experimental Data).
3.10
Universal dimensionless mean velocity profile of turbulent flow close to a smooth wall according to the data of tube-, channel- and boundary-layer measurements. Here $u_+ = u/u_*$ and $z_+ = u_*/v$. (From Monin and Yaglom 1971)