Constitutive Equations for a Newtonian Fluid

Learning Objectives:
1. List and apply the basic assumptions used in classical fluid dynamics for ocean engineering
2. Interpret the physical meaning of different terms in the deformation tensor, including dilatation, shear strain, and rotation

Topics/Outline:
1. Definition of a Newtonian Fluid
2. Derivation of the stress tensor relation for a Newtonian fluid
3. Discussion of the viscosity coefficients

Reading:
This lecture is derived from Currie (2003):

Chapter 1.

You may also consult lecture notes by Mei (2002):

Lecture 1.4: Forces in the Fluid
Lecture 1.6: Relations between Stress and Rate-of-Strain Tensors
Available online at:

Many other texts have very good discussion of the stress tensor and Newtonian fluids.
Stress Tensor for a Newtonian Fluid:

The momentum equation contains the unknown surface stress tensor $\sigma_{ij}$.

We seek an expression relating $\sigma_{ij}$ to other fluid properties.

**Constitutive Equation:** (Wikipedia) A relation between two or more physical quantities that is specific to a material or substance and approximates the response of that material to external forces.

**Newtonian Fluid:** An idealized fluid that approximates the behavior of water, air and many other fluids.

The constitutive equation for a Newtonian fluid satisfies the following conditions. (Currie, p. 27)

1. When at rest, the stress is **hydrostatic** and the pressure in the fluid is the thermodynamic pressure.

2. The stress tensor $\sigma_{ij}$ is **linearly proportional** to $\dot{e}_{ij}$ (the deformation tensor).

3. No shear stresses may act during **solid body rotation**.

4. There are no preferred directions in the fluid, so the fluid properties are point properties.

We seek the most general equation satisfying these properties.
1.) Apply constraint 1: $\sigma_{ij} = -p$ when fluid is at rest.

$\sigma_{ij} = -p \delta_{ij} + \tau_{ij}$

thermodynamic pressure $p$.
(-) since sign convention is that
(+) $\sigma_{ij}$ is tension.

Must be zero when fluid is at rest.
$\tau_{ij}$ called the shear stress tensor.

$\delta_{ij}$ selects only the normal components of the stress tensor $\rightarrow p$ acts normal to a surface.

$\sigma_{ij} = 0$, if $i \neq j$; recall definition of a fluid as a substance that moves under any non-zero shear stress.

2.) Apply constraint 2: $\sigma_{ij} \propto D_{kl}$

$T_{ij}$ has nine components, each of which is linearly proportional to $D_{kl}$.
$\rightarrow 9 \cdot 9 = 81$ proportionality constants $\alpha$

$\tau_{ij} = \alpha_{ijk} \frac{\partial u_k}{\partial x_i}$

Note double summation.
$\alpha_{ijk}$ is a $4^{th}$ order tensor of $81$ proportionality constants.
3. Apply Constraint 3: shear stress is zero for solid body motion.

\[ \frac{\partial u_k}{\partial x_l} = \epsilon_{k \ell} + \omega_{k \ell} \]

Rotation tensor is non-zero for solid body rotation. Non-zero for shearing motion, but is zero for solid body rotation.

Thus, \( T_{ij} \) must only be proportional to symmetric part of \( \frac{\partial u_k}{\partial x_l} \).

\[ T_{ij} = \beta_{ij \ell} \left( \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \right) \]

Still 81 proportionality constants, they're just different from \( \alpha_{ij \ell} \).

4. Apply Constraint 4: No preferred direction.

First, what does this mean? What if there is stratification. Think of a submarine traveling at a fixed speed \( V \):

\[ \text{always the same.} \]

It means that the magnitude of the resistance (friction, pressure drag) is the same for any direction of motion.

This property is called "isotropy", and \( \beta_{ij \ell} \) must obey it.

We need an isotropic tensor of rank 4.
Isotropic Tensors:

<table>
<thead>
<tr>
<th>Rank</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.</td>
<td>Scalars: all zero-order tensors are isotropic → they have no direction.</td>
</tr>
<tr>
<td>1.</td>
<td>Vectors: none are isotropic. All have direction.</td>
</tr>
</tbody>
</table>
| 2.   | One isotropic tensor: 
\[ \alpha \delta_{ij} \] is isotropic 
\[ = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \] keeps the same values if we rotate the axes. |
| 3.   | Like vectors, there are no isotropic 3rd-order tensors. |
| 4.   | Most general 4th-order isotropic tensor is: 
\[ \beta_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \] + \delta (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) |

Because \( T_{ij} \) is symmetric (see step 3 above) the contribution from the \( \beta \) term will be zero.

\[
T_{ij} = \left[ \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) \\
= \lambda \delta_{ij} \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) + \mu \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
= \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) 
\]
Combining:
The stress tensor for a Newtonian fluid is

\[
\sigma_{ij} = -p\delta_{ij} + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

\(\lambda, \mu\): scalar viscosity coefficients found by experiment.

\(\mu\): dynamic viscosity:
consider Couette flow

\[
\begin{align*}
\sigma_{11} &= -\rho + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\
&= -\rho \\
\sigma_{22} &= -\rho \\
\sigma_{33} &= -\rho \\
\sigma_{12} &= \sigma_{12} = \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \mu \frac{\partial u}{\partial y} = \tau_{12} = \tau_{21} \\
\sigma_{31} &= \sigma_{31} = \sigma_{23} = \sigma_{32} = 0
\end{align*}
\]

Newton's law of viscosity

\(\tau = \mu \frac{\partial u}{\partial y}\) in simple shear flow,

\(\mu\): dynamic viscosity.

\(\nu = \frac{\mu}{\rho}\): kinematic viscosity.
\( \lambda \): 2\textsuperscript{nd} viscosity coefficient.

Compare the thermodynamic pressure to the normal stresses:

\[
\overline{P} = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad \text{(called the mechanical pressure)}
\]

\[
= \frac{1}{3} \left( -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu \frac{\partial u}{\partial x} - p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu \frac{\partial v}{\partial y} 
\right)
\]

\[
= -p + \lambda \frac{\partial u_k}{\partial x_k} + \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
= -p + \left( \lambda + \frac{2}{3} \mu \right) \frac{\partial u_k}{\partial x_k}
\]

\( K \): bulk viscosity.

1) Incompressible fluid: \( \frac{\partial u_k}{\partial x_k} = 0 \rightarrow \overline{P} = p \)

Thermodynamic and mechanical pressure are equal.

It doesn't matter what \( \lambda \) or \( K \) are since \( \frac{\partial u_k}{\partial x_k} = 0 \) and their terms make no contribution to the stress.

2) Monoatomic perfect gas: \( K = 0 \rightarrow \lambda = -\frac{2}{3} \mu \)

Stokes relation: \( \lambda = -\frac{2}{3} \mu \).

3) Polyatomic gases and liquids:

compare \( p \) and \( \overline{P} \):
Molecular energy includes:

\[
\text{translation } = \overline{p} \\
+ \text{ rotation } \\
+ \text{ vibration } \\
+ \text{ intermolecular attraction }
\]

\[
= p
\]

\[p - \overline{p}\] is usually small.

\[k \frac{du_k}{dx_k} = \text{transfer of mechanical energy into other forms of molecular energy}\]

Since departure from \(k=0\) is small, using Stokes relation is usually acceptable, and is only necessary if \(\frac{du_k}{dx_k} \neq 0\).