Stokes First Problem

Learning Objectives:
1. Write the exact equations for a fluid flow problems incorporating applicable simplifications
2. List and explain the assumptions behind the classical equations of fluid dynamics

Topics/Outline:
1. Define Stokes first problem
2. Apply similarity solution method to Stokes first problem
3. Solve the ODE for the similarity function

Reading:
Chapter 7
Chapter 6
Chapter 9
Stokes First Problem:
Flow near an impulsively started plate.

\[ \begin{align*} \text{Fluid initially at rest.} \\
\rightarrow U \text{ suddenly imposed at } t=0. \\
\text{Infinite plate in } xz-\text{plane.} \end{align*} \]

Boundary Conditions:

- Impulsive start at \( y=0 \):
  \[ u(0) = \begin{cases} U & t > 0 \\ 0 & t \leq 0 \end{cases} \]

  (Because the whole water column will need time to adjust, the flow is transient.)

  \[ \frac{\partial u}{\partial t} \neq 0. \]

- Fluid at rest at \( y = \infty \):
  The whole water column is at rest before
  the plate starts to move.
  The effect of the moving plate will never be
  felt at \( y = \infty \) because the plate is infinitely far
  away.
  Hence, the fluid at \( y = \infty \) will never accelerate:
  \[ u(\infty) = 0. \]

Initial Condition:
Fluid in the half-plane \( y > 0 \) is initially
at rest:
\[ u(y \geq 0, t=0) = 0. \]
Simplify the governing Navier-Stokes Equations:

**Conservation of Mass:**

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

- \( u \) uniform in \( x \)-direction.
- \( v = 0 \) at \( y = 0 \).
- \( w = 0 \) at \( z \rightarrow \).

**Conclusions:**

- \( u \neq f(x) \)
- \( v = 0 \)
- \( w = 0 \)

**Y-direction Conservation of Momentum:**

With \( v = 0 \):

\[ \frac{\partial p}{\partial y} = 0 \rightarrow p \neq f(y) \]

**Z-direction Conservation of Momentum:**

With \( w = 0 \):

\[ \frac{\partial p}{\partial z} = 0 \rightarrow p \neq f(z) \]

\[ \therefore p \text{ can be a function of } x \text{ only:} \]

\[ \frac{\partial p}{\partial x} = \frac{dp}{dx} \]
**X-direction Conservation of Momentum:**

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \sqrt{\frac{\partial u}{\partial y}} + \nu \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\( u \): uniform and \( \infty \) in \( x \)-direction.

\( \rho \): uniform and \( \infty \) in \( y \)-direction.

\( \nu \): uniform and \( \infty \) in \( z \)-direction.

\[
\frac{\partial u}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

Assume no external pressure gradient. At \( t \leq 0 \) if fluid is at rest.

**Mathematical Formulation:**

Thus, we must solve:

\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}
\]

\( u(y, 0) = 0 \) for \( y > 0 \)

\( u(0, t) = \frac{\nu}{\nu} \) for \( t > 0 \)

\( 0 \) for \( t \leq 0 \)

\( u(\omega, t) = 0 \)

**Expected Solution:**

[Graph showing increasing time]
Similarity Solution Method:

Because the plate is infinite and the fluid domain is semi-infinite, there is no geometric length-scale imposed on the solution. Thus, dimensional analysis gives:

\[ u = f \left( \frac{v}{y}, y, t \right) \]

only forms one independent nondimensional number:

\[ \eta = \frac{y}{\sqrt{v t}} \]

And we expect all velocity profiles to collapse to a single shape:

"Similar" because all profiles have the same basic shape.

So, we assume the solution:

\[ u = f(\eta = \frac{y}{\sqrt{v t}}) \]

and substitute into the differential equation to find the unknown function \( f(\eta) \).
Solution:

Substitute similarity solution:
\[
\frac{\partial}{\partial t} \left( \nu f \left( \frac{y}{\sqrt{\nu t}} \right) \right) = \nu \frac{\partial}{\partial y} \left( \frac{2}{\nu^2} \left[ \nu f \left( \frac{y}{\sqrt{\nu t}} \right) \right] \right)
\]

+ chain rule
\[
\nu \frac{\partial^2 f}{\partial y \partial t} = \nu \nu \frac{\partial^2 f}{\partial y^2} \left( \frac{\partial^2 y^2}{\partial y^2} \right)
\]

Needed:
\[
\frac{\partial y}{\partial t} = \frac{2}{\nu t} \left( \frac{y}{\nu t^{1/2}} \right) = \frac{y}{\nu t} \left( \frac{1}{2} \right)^{-3/2}
\]
\[
= -\frac{y}{2\sqrt{\nu t}} \frac{1}{t}
\]
\[
= \frac{-\frac{y}{2t}}{
\]
\[
= \frac{\partial y}{\partial t} = \frac{2}{\nu t} \left( \frac{y}{\nu t^{1/2}} \right) = \frac{1}{\sqrt{\nu t}}
\]

Substituting:
\[
\gamma \frac{\partial f}{\partial y} \left( -\frac{y}{2t} \right) = \gamma \gamma \frac{\partial^2 f}{\partial y^2} \left( \frac{1}{\nu t} \right)^2
\]
\[
\frac{d^2 f}{d y^2} + \frac{\gamma}{\nu} \frac{df}{dy} = 0
\]
Initial Condition:
- At \( t = 0 \):
  \[
  \nu f \left( \frac{y}{\sqrt{\nu t}} \right) = 0 \quad y = 0
  \]
  \[f(\infty) = 0\]

Boundary Conditions:
- At \( y = 0 \):
  \[
  \nu f \left( \frac{0}{\sqrt{\nu t}} \right) = \nu
  \]
  \[f(0) = 1\]
- At \( y = \infty \):
  \[
  \nu f \left( \frac{\infty}{\sqrt{\nu t}} \right) = 0 \quad t = 0
  \]
  \[f(\infty) = 0\]

Thus, we must solve:
\[
\frac{d^2f}{d\eta^2} + \frac{1}{2} \frac{df}{d\eta} = 0
\]
\[\quad f(0) = 1\]
\[f(\infty) = 0\]

to solve this, note:
\[
\frac{d}{d\eta} \left( \ln \frac{df}{d\eta} \right) = \frac{1}{df/d\eta} \cdot \frac{d^2f}{d\eta^2}
\]

Thus, we can re-write the ODE as:
\[
\frac{d}{d\eta} \left( \ln \frac{df}{d\eta} \right) = -\frac{\eta}{2}
\]
Integrating:
\[ \int \frac{df}{d\eta} (\ln \frac{df}{d\eta}) d\eta = \int -\frac{\eta}{2} d\eta \]
\[ \ln \frac{df}{d\eta} = -\frac{\eta^2}{4} + C \]

Take exponential of both sides:
\[ \frac{df}{d\eta} = C_0 \exp(-\frac{\eta^2}{4}) \]

Integrating again:
\[ \int \frac{df}{d\eta} d\eta = \int C_0 \exp(-\frac{\eta^2}{4}) d\eta \]
\[ f(\eta) = C_0 \int_0^\eta \exp(-\frac{x^2}{4}) dx + C_1 \]

Apply B.C. \( f(0) = 1 \):
\[ f(0) = C_0 \int_0^0 \exp(-\frac{x^2}{4}) dx + C_1 = 1 \]
\[ C_1 = 1 \]

Apply B.C. \( f(\infty) = 0 \):
\[ f(\infty) = C_0 \int_0^\infty \exp(-\frac{x^2}{4}) dx + 1 \]

\[ \Rightarrow \text{Change } \left(\frac{x}{2}\right)^2 \to \frac{s^2}{2} \]
\[ \frac{x}{2} = 2s \]
\[ ds = 2ds \]
\[ f(\infty) = C_0 \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \, dx + 1 \]

\[ = 2C_0 \sqrt{\frac{\pi}{2}} + 1 \]

\[ C_0 = -\frac{1}{\sqrt{\pi}} \]

Collecting the solution:

\[ f(\eta) = -\frac{1}{\sqrt{\pi}} \int_0^{\eta} \exp\left(-\frac{\xi^2}{4}\right) \, d\xi + 1 \]

\[ \rightarrow \text{Change } \left(\frac{\xi}{2}\right)^2 \text{ to } \eta^2 \]

\[ \xi = 2\eta \quad \rightarrow \quad \eta = \frac{\xi}{2} \]

\[ d\xi = 2 \, d\eta \]

\[ = -\frac{2}{\sqrt{\pi}} \int_0^{\eta/2} \exp\left(-\eta^2\right) \, d\eta + 1 \]

\[ = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta/2} \exp\left(-\eta^2\right) \, d\eta \]

Called the error function, arises during integration of the Gaussian probability density function.

\[ = 1 - \text{erf} \left(\frac{y}{2\sqrt{Dt}}\right) \]

Thus

\[ \frac{u}{v} = 1 - \text{erf} \left(\frac{y}{2\sqrt{Dt}}\right) \]