Von Karman’s Momentum Integral

Learning Objectives:

1. Develop approximations to the exact solution by eliminating negligible contributions to the solution using scale analysis
2. Identify and formulate the physical interpretation of the mathematical terms in solutions to fluid dynamics problems

Topics/Outline:

1. Momentum integral concept
2. Derivation of von Karman’s momentum integral
3. Application of the momentum integral to fitting of the sin() function to the Blasius boundary layer

Reading:

Chapter 9

Chapter 10

Chapter 3

Chapter 12
von Karman's Momentum Integral.

In the last lecture, we saw that the boundary layer thickness ($\delta$) was related to the shear stress distribution on the plate ($\tau_0$).

This can be explained as follows:

$\delta$ grows with distance due to diffusion (by $\delta$) of momentum deficit ($u(x, 0) = 0$) from the plate surface.

For a thin boundary layer ($\delta x \to 0$) the von Karman Momentum integral shows that:

"the rate of change of the momentum in the entire boundary layer [i.e. integrated over the height $\delta$] at any value of $x$ is equal to the force produced by the shear stress at the surface at that location."

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This is useful, e.g., if we want to fit simple analytical profiles of the velocity while maintaining proper relationships among $\delta$, $\Theta$, $\delta^*$, and $\tau_0$.

Example: $u(y) = a_1 \sin(a_2 y)$

$\Rightarrow$ what are the best values of $a_1$ and $a_2$?

$\Rightarrow$ especially useful if $\frac{d\tau_0}{dx} \neq 0$. 

General Boundary Layer Equations:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \]

\[ u(x,0) = v(x,0) = 0 \]

\[ u(x,\infty) = U(x) \]

We saw before that we can use Bernoulli’s Equation in the outer flow:

\[ \frac{d}{dx} \left( \frac{p + \frac{u^2}{2} + \rho_0 \frac{v^2}{2}}{\rho} \right) = \text{const.} \]

Streamline in outer flow is along a constant value of \( z \):

\[ \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{d^2 U}{dx^2} = 0 \]

\[ \nu \frac{dU}{dx} = -\frac{1}{\rho} \frac{dp}{dx} \quad \text{→ substitute into momentum equation.} \]

Conservation of Mass:
Use to relate \( v \) to \( u \):

\[ \int \frac{\partial v}{\partial y} \, dy = \int -\frac{\partial u}{\partial x} \, dy \]

\[ v(y) = -\int_0^y \frac{\partial u}{\partial x} \, dy' \]
Conservation of Momentum:

Substitute from conservation of mass and integrate over the boundary layer to a height $h$:

$$\int_0^h \{ u \frac{du}{dx} + \frac{\partial u}{\partial y} \int_0^y - \frac{\partial u}{\partial x} \ dy \} \ dy = \int_0^h v \frac{\partial u}{\partial y} \ dy$$

Like taking a boundary layer average. Outer pressure term.

RHS:
$$\int_0^h v \frac{\partial u}{\partial y} \ dy = v \frac{\partial u}{\partial y} \bigg|_h$$

$$= v \frac{\partial u}{\partial y} \bigg|_h - v \frac{\partial u}{\partial y} \bigg|_0$$

Zero at the top of the b.l.

Newton's law of viscosity:
$$\tau_0 = \mu \frac{\partial u}{\partial y} \bigg|_0$$

LHS, 2nd term:
$$\int_0^h \frac{\partial u}{\partial y} \int_0^y - \frac{\partial u}{\partial x} \ dy \ dy$$

Integrate by parts: $\int u dv = uv - \int v du$

$u = \int_0^h - \frac{\partial u}{\partial x} \ dy \bigg|_0$
$dv = \frac{\partial u}{\partial y} \ dy$

$du = - \frac{\partial u}{\partial x} \ dy$
$v = u \bigg|_0^h = U(x) - \varphi$

$$= U(x) \int_0^h - \frac{\partial u}{\partial x} \ dy - \int_0^h - u \frac{\partial u}{\partial x} \ dy$$
Substituting:
\[
\int_0^h \left\{ u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial x} \right\} \, dy = -\frac{\tau_0}{\delta}
\]
\[\uparrow \text{ add and subtract } u \frac{\partial u}{\partial x} \quad \text{flip sides to reverse signs.}\]

We may collect terms on the RHS to obtain:
\[
\int_0^h \frac{\partial}{\partial x} (u (u - u)) \, dy + \frac{dV}{dx} \int_0^h (u - u) \, dy = \frac{\tau_0}{\delta}
\]
\[\downarrow \text{ contribution is negligible above } h \to \text{ set limits to } \infty.\]

\[
\int_0^\infty \frac{\partial}{\partial x} (u (u - u)) \, dy + \frac{dV}{dx} \int_0^\infty (u - u) \, dy = \frac{\tau_0}{\delta}
\]
\[\downarrow \text{ since limits are independent of } x \text{; derivative become ordinary since integral will be independent of } y.\]

\[
\frac{d}{dx} \int_0^\infty u (u - u) \, dy + \frac{dV}{dx} \int_0^\infty (u - u) \, dy = \frac{\tau_0}{\delta}
\]
\[\begin{align*}
\equiv & \quad u^2 \theta \\
= & \quad \delta^* V
\end{align*}
\]

Hence, the momentum equation becomes:

\[
\frac{\tau_0}{\delta} = \frac{d}{dx} (u^2 \theta) + \delta^* V \frac{dV}{dx}
\]
\[\uparrow \text{ relates } \tau_0 \text{ to } \theta \text{ and } \delta^* \text{ for given } V(x).\]
Application: fit analytical functions to match correct behavior of $x, \theta, \delta^*$ in boundary layer.

Fit $\sin(.)$ to velocity profile.

1.) Fit equation to boundary and match conditions.

![Graph: $u$ vs $y$, $\delta$ as a function of $u(x)$]

**Conditions to apply in order of importance:**

\[
u(x,0) = 0 \]
\[
u(x,\delta) = \nu(x) \]
\[
\frac{\partial u}{\partial y} \bigg|_{x,\delta} = 0
\]
\[
\frac{\partial^2 u}{\partial y^2} \bigg|_{x,0} = 0
\]
\[
\frac{\partial^2 u}{\partial y^2} \bigg|_{x,\delta} = 0
\]

Fit $u = a_1 \sin(a_2 y)$:

\[
u(x,0) = 0 \rightarrow \text{satisfied automatically}
\]
\[
u(x,\delta) = \nu(x) = a_1 \sin(a_2 \delta)
\]
\[
\nu'(x,\delta) = 0 = a_1 \cos(a_2 \delta) a_2
\]
\[
\Rightarrow a_2 = \frac{\pi}{2\delta}
\]
\[
\Rightarrow \text{then condition 2 gives}
\]
\[
a_1 = \nu(x)
\]

Thus, we want to fit:

\[
u(x,y) = \nu(x) \sin \left( \frac{\pi y}{2\delta} \right)
\]

(use momentum integral to find $\delta$.)
2) Evaluate $\delta(x)$ from momentum integral:

$$\frac{\tau_0}{\delta} = \frac{d}{dx} (U^2 \theta) + \delta^* U \frac{dU}{dx}$$

Find $\theta$, $\delta^*$, $\tau_0$.

$$\theta = \int_0^\infty \frac{U}{U} \left(1 - \frac{U}{U}ight) dy = \int_0^\infty \frac{U \sin \left(\frac{\pi y}{2\delta}\right)}{U} \left(1 - \frac{\sin \left(\frac{\pi y}{2\delta}\right)}{U}\right) dy$$

$$= \frac{4-\pi}{2\pi} \delta$$

$$\delta^* = \int_0^\infty \left(1 - \frac{U}{U}\right) dy = \int_0^\infty \left(1 - \frac{\sin \left(\frac{\pi y}{2\delta}\right)}{U}\right) dy$$

$$= \frac{\pi - 2}{\pi} \delta$$

$$\tau_0 = \frac{dU}{dy} \Big|_{y=0} = \mu \left[U(x) \cos \left(\frac{\pi y}{2\delta}\right) \frac{\pi}{2\delta}\right]_{y=0}$$

$$= \frac{\mu U \pi}{2\delta}$$

Substituting:

$$\frac{U \pi}{2\delta} = \frac{d}{dx} \left(U^2 \left(\frac{4-\pi}{2\pi} \delta\right)\right) + \left(\frac{\pi - 2}{\pi}\right) \delta U \frac{dU}{dx}$$

To compare to Blasius solution, let $U = \text{constant}$.

$$\frac{U \pi}{2\delta} = \frac{4-\pi}{2\pi} U \frac{d\delta}{dx}$$

with initial condition $\delta(0) = 0$
\[
\frac{\pi^2}{(4-\pi)} \frac{\sqrt{U}}{V} = 8 \frac{dx}{dx}
\]

\[
\int \delta d\delta = \left( \frac{\pi^2}{(4-\pi)} \frac{\sqrt{V}}{U} \right) dx
\]

\[
\frac{\delta^2}{2} = \frac{\pi^2 V}{(4-\pi)U} x + C
\]

\[
\delta = \sqrt{\frac{2\pi^2 \frac{V}{4-\pi} - \frac{Vx}{U}}}{x}
\]

\[
\frac{\delta}{x} = \sqrt{\frac{2\pi^2}{4-\pi} \frac{1}{Re_x}}
\]

\[
\frac{\delta}{x} = \frac{4.795}{\sqrt{Re_x}}
\]

Advantage is that this method can accommodate \(V(x)\).

\[
\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}
\]

Compare to from numerical solution.